CS 559: Algorithmic Aspects of Computer Networks Spring 2010

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18.1 Erdos-Renyi Random Graphs

An Erdos-Renyi graph $G(n, p)$ depends on two parameters n and p s.t. $n =$ number of nodes and $p =$ probability a given edge (i, j) is present

18.1.1 Small p

- If p is small $\sim \frac{1}{n}$, then not every node will be connected. Many components will be disconnected from the graph.
- The average degree will be constant (disconnected forests).
- In general, these networks are not very interesting.

18.1.2 Larger p

If $p \sim \frac{log(n)}{n}$...

- The average degree will be $\sim log(n)$ per node
- Nodes being disconnected is unlikely (inverse polynomial of n), some singletons but not many.
- Most nodes will have a high degree

If $p \sim \frac{2*log(n)}{n}$...

- Sharp phase transition, the network gels and the graphs becomes one giant connected component

18.1.3 Degree distribution

The Erdos-Renyi graph degree distribution resembles a normal distribution where the degrees of most nodes are about the same size.

$Pr[X \geq x] \leq e^{-x}$. The graph is characterized by exponential tails.

18.2 S-W Graphs

Some social scientists and physicists claimed Erdos-Renyi graphs were not the types of graphs they usually see in practice [1][2]. They saw distributions tailing off over many orders of magnitude with many high degree nodes. This is termed a "heavy-tailed" degree distribution. The probability drops off sub-exponentially, where $Pr[X \geq x] \leq \frac{1}{poly(x)}$

Strogatz-Watts graphs vary a parameter p which varies from 0 to 1. Values towards 0 represent regularity whereas those towards 1 represent disorder.

In the extreme, regularity creates a basic topology like a ring. There is an additional parameter k which controls the connectivity of the graph.

The model focuses on two computable quantities:

- $L(p)$ = average path length
- $C(p)$ = average clustering coefficient

The clustering coefficient is a measure of the "connectedness" of a neighborhood. The clustering coefficient is calculated as follows for a node x. Of the neighbors of x, calculate the fraction of neighbors $N(x)$ that are directly connected (pairwise acquaintances). It is a good idea to normalize by dividing by the total number of possible edges:

$$C(p,x) = \frac{\|[(y,z) : y \in N(x), z \in N(x)]\|}{\binom{N(x)}{2}}$$

$$C(p) = \frac{\sum_x C(p,x)}{n}$$

Scaling k up, results in better clustering.

18.2.1 Generating graph

- Take a model which starts at full regularity then do "random rewiring".
- Iterate around ring and for every edge rewire the edge with probability p
- Rewirings to edges that already existed are ignored
- For each edge (i,j) in turn, rewire the edge to (i,k) for some $k \in_R \{1..n\}$ with prob. p and only if $(i,k) \notin E$

At p = 1 the graph is very similar to a Erdos-Renyi graph, random rewiring shortens path lengths since long jumps shorten paths, but this reduces the clustering coefficient.

When p is in the middle, the average path length is smaller and clustering coefficient is higher.

Strogatz and Watts argue that "real" graphs are not too far from $p = 0$.

18.3 Graph Models

	Erdos Renyi	Strogatz-Watts	"Internet"
Path Length	$p \sim \frac{\log n}{n} \implies O(\log n)$	$O(n) \to O(\log n)$	$O(\log n)$
Clustering Coefficient	no better than random	High clustering, decreases w/ p	> random
Degree Distributions	exponential tails, normal	exponential tails	subexponential tails, heavy tailed

Table 18.1. Random graph models

Bibliography

[1] Barabasi, A. L., and Albert, R. Emergence of scaling in random networks. *Science 286*, 5439 (1999), 509–512.

[2] Watts, D. J., and Strogatz, S. H. Collective dynamics of /‘small-world/’ networks. *Nature 393*, 6684 (jun 1998), 440–442.