Reduction Example - Giving a reduction from $A_{TM}$ to $E$

In this example we consider how to prove the language $E = \{<M> | L(M) = \phi\}$ is not decidable. Recall $\phi$ = the empty set.

Here $L(M)$ is the collection of inputs that $M$ accepts and so $E$ is the set of all algorithms (or TM's) which do not accept any of their inputs.

To prove $E$ undecidable we use a reduction showing that $A_{TM} \leq E$. (That is, $A_{TM}$ is computably reducible to $E$.) Since we have already proved that $A_{TM}$ is undecidable, this reduction implies that $E$ is also undecidable.

So how do we prove that $A_{TM} \leq E$? Following the definition of a reduction, we ASSUME that there is an algorithm $E$ which decides the language $E$ above.

Then our GOAL is to use find an algorithms $A$ which uses the assumed algorithms $E$ as a subroutine and which decides $A_{TM}$.

So let’s start by explaining our assumption that $E$ exists. $E$ decides $E$, this means $E$ takes some $<M>$ as input, computes this input $<M>$ and always halts. $E (<M>)$ accepts if and only if $<M>$ is in $E$ which is true exactly when $L(M) = \phi$, the empty set.

Now given an input $(M,w)$ to the $A_{TM}$ problem (or language) we first construct an algorithm $M'$ from $M$ as follows:

1. $M'$ will have the same input alphabet as $M$.
2. For any input $z$ to $M'$, $M'(z)$ first checks if $z = w$.
3. If $z = w$ then $M'(z)$ simply simulates $M(w)$ and does whatever does (accepts, rejects or loops).
4. If $z \neq w$ then $M'(z)$ halts and rejects $z$.

So in summary, given $(M,w)$, the $M'$ we construst either accepts no strings at all or it accepts the string $w$.

And $(M,w)$ is in $A_{TM}$ if and only if the $M'$ has $L(M') = \{w\}$.
Now it is easy to state the algorithm $A$. Here is what $A$ does on an input $(M,w)$.

1. First we use $(M,w)$ to construct the machine $M'$ as above.

2. Now use the algorithm $E$ which we assume exists. Input $<M'>$ to $E$.
As $E$ is a decider $E(<M'>)$ will halt and either accept or reject. This tells us whether $L(M')$ is non-empty or empty.

3. If $L(M')$ is non-empty then we know that $M(w)$ accepts and so we have $A$ halt and accept.
If $L(M')$ is empty then we know that $M(w)$ does not accept, so we halt $A$ and it rejects.

Hence we have $(M,w) \in A_{TM}$ if and only if $A$ accepts, and so $A$ decides the language $A_{TM}$, contradicting its undecidability.