CAS CS 113. Solution to Problem Set 2

Solution 7. We wish to prove that the Odometer Principle (OP) allows a mechanical device to count correctly in base-b. More formally, we wand to show that for all $n \in \mathbb{N}$ the following proposition is true:

Proposition $P(n)$**:** After n mechanical increments according to the OP, an odometer that initially displayed a single rightmost 0 and all blanks to its left, displays the base-b representation of the number n.

Proof. We prove the validity of the proposition for all $n \in \mathbb{N}$ by induction[1].

Base Case: (n=0) The odometer has its initial setting so $P(0)$ is trivially true[2].

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$. That is, after k mechanical increments the odometer shows the base b representation of k.

Induction Step: We must show that after $k+1$ executions of the OP the odometer displays the base-b representation of $k+1$.

By the induction hypothesis, after k mechanical increments the odometer is correct and displays the base-b representation of k. Let $x_{m-1}x_{m-2}\ldots x_1x_0$ denote the non-blank digits of this representation so that $k = \sum_{i=0}^{m-1} x_i b^i$ where $m = \lceil \log_b k+1 \rceil$ and $x_i \in \{0, 1, \ldots, b-1\}$ for all $0 \leq i < m$. Now consider the effect of the $(k+1)$st mechanical increment on this setting. The are three cases[3].

Case 1 If $x_0 < b-1$, Step 1 of the OP changes the rightmost digit to $x'_0 = x_0 + 1$ and leaves all other digits unchanged. Since adding one to any base b number whose last digit is not $b-1$ produces the same result, $P(k+1)$ is true in this case.

Case 2 If the $0 < j < m$ rightmost digits are $b-1$ and $x_j \neq b-1$, Step 3 is repeatedly executed until x_j is reached at which point Step 1 is executed. The resulting odometer reads $x_{m-1}x_{m-2}\ldots x'_j \underbrace{00\ldots0}_{j \text{ zeros}}$ where $x'_j = x_j + 1$. Again, this corresponds to the case when one is added to a base-b number k of the form $x_{m-1}x_{m-2}\ldots x_j \underbrace{(b-1)(b-1)\ldots(b-1)}_{j\ (b-1)'\text{s}}$.

Case 3 If for all $0 \leq i < m$ is it the case that $x_i = b-1$, that is $k = b^m - 1$, Step 3 is executed m times followed by an execution of Step 2. The resulting odometer reads $1\underbrace{00\ldots0}_{m \text{ zeros}}$, i.e. a 1 followed by m zeroes which is just b^m and again correct.

Thus, $P(k+1)$ is true in all three cases. Therefore, we conclude that the Odometer Principle counts correctly in base-b.

□

[1] The proof using strong induction that Scott discussed with several people during his office hours, is perhaps more complicated than this proof.

[2] Alternatively, after $n = 1$ execution of the OP, the rightmost digit of the odometer shows 1 and all other digits remain blank. Hence, $P(1)$ is true.

[3] In fact the first two cases can be combined into a single case leaving only two cases.