Laplace Transforms

- Formally, Laplace transform for a signal $f(t)$ is:
  \[ F(s) = \int_0^\infty f(t)e^{-st}dt \]
  \[ s = \sigma + j\omega \]

- The Power: Ability to study linear systems using algebraic equations

  - Example:
    \[ a_1 y(t)+a_2 y(t)+a_3 y(t) = h_1 u(t)+h_2 u(t) \]
    \[ \Rightarrow Y(s) = \frac{h_1 s + h_2}{a_1 s^2 + a_2 s + a_3} \cdot U(s) \]

Laplace Transforms

- Basic translations
  - Impulse signal $f(t)=\delta(t) \Leftrightarrow F(s)=1$
  - Step signal $f(t)=a_1(t) \Leftrightarrow F(s)=\frac{a}{s}$
  - Ramp signal $f(t)=a_2 t \Leftrightarrow F(s)=\frac{a}{s^2}$
  - Exp signal $f(t)=e^{at} \Leftrightarrow F(s)=\frac{1}{s-a}$
  - Sinusoid signal $f(t)=\sin(at) \Leftrightarrow F(s)=\frac{a}{s^2+a^2}$

- Composition rules
  - Linearity $L[a f(t)+b g(t)]=a L[f(t)]+b L[g(t)]$
  - Differentiation $L[\frac{df(t)}{dt}]=sF(s)-f(0)$ if $f(0)=0$
  - Integration $L[\int_0^t f(\tau)d\tau]=\frac{F(s)}{s}$
  - Convolution $y(t) = g(t)*u(t) = \int_0^t g(t-\tau)u(\tau)d\tau \quad Y(s) = G(s)U(s)$

Vegas-like Model

- Error: $e(t)=B_r - b(t)$
- Model (differential equation):
  \[ \frac{d}{dt} b(t) = w(t) - d(t) \]
- Controller $C$? $e(t) \Rightarrow w(t)$
Vegas-like Model

- Error: \( e(t) = B_r - b(t) \)
- Model (differential equation): \( \frac{d}{dt}b(t) = w(t) - d(t) \)
- Controller \( C \): \( e(t) = w(t) \)

\[
C(t) \quad B_r \\
\downarrow \\
\text{Queue} \\
\downarrow \\
\ast(d(t)) \\
\ast(b(t))
\]

Block Diagram

- A pictorial tool to represent a system based on transfer functions and signal flows
- Represent a feedback control system

\[
\begin{align*}
R(s) & \quad C(s) \quad G_c(s) \quad Y(s) \\
\downarrow & \quad \downarrow & \quad \downarrow & \quad \downarrow \\
R(s) & \quad G_o(s) \quad Y(s) \\
\end{align*}
\]

\[G_c(s) = \frac{C(s)G_o(s)}{1 + C(s)G_o(s)} \]

\[Y(s) = G_c(s)R(s)\]

Vegas-like Model

- Transfer Function & Block Diagram

- Buffer occupancy is modeled as a differential equation
  \[
  \frac{d}{dt}b(t) = w(t) - d(t) + \epsilon(t)
  \]
- Inputs: reference \( B_r(s) \); service rate \( D(s) \)
- Closed-loop system transfer functions
  - \( B(s) \) as input: \( T_b(s) = \frac{C(s)G_o(s)}{1 + C(s)G_o(s)} \)
  - \( D(s) \) as input: \( T_d(s) = -\frac{G_o(s)}{1 + C(s)G_o(s)} \)
- Output: \( B(s) = T_b(s)B_r(s) + T_d(s)D(s) \)
**Proportional Control**

**Stability of Vegas-like Control**

- **Proportional Controller**
  - \( C(s) = K \)
- **Transfer functions**
  - \( B(s) \) as input: \( T_1(s) = \frac{K}{s+K} \)
  - \( D(s) \) as input: \( T_2(s) = -\frac{1}{s+K} \)
- **Stability**
  - Pole = \(-K\) \iff System is stable for \( K > 0 \)

![Transfer functions diagram]


**Vegas-like Model (PI controller)**

- **Error:** \( e(t) = B - b(t) \)
- **Model (differential equation):** \( \frac{d}{dt} b(t) = w(t) - d(t) \)
- **Controller** \( C(s) = \frac{K}{s} \)

![Model diagram]


**Integral Control**

**Stability of Vegas-like Control**

- **Buffer occupancy is modeled as a differential equation**
  \[
  \frac{d}{dt} w(t) = w(t) - d(t) + \frac{W(s) - D(s)}{s} \]
- **Integral Controller**
  - \( \frac{d}{dt} w(t) = K_0(t) \)
  - \( C(s) = \frac{K}{s} \)
- **Transfer functions**
  - \( B(s) \) as input: \( T_1(s) = \frac{K}{s^2+K} \)
  - \( D(s) \) as input: \( T_2(s) = -\frac{s}{s^2+K} \)
- **Stability**
  - Pole = \( \pm jK \) \iff System is critically stable

![Integral control diagram]
Performance Specifications

Stability

A linear time-invariant system is stable if all poles of its transfer function are in the left-hand side of the s-plane ($\forall p_i, \text{Re}[p_i] < 0$)

$$G(s) = \frac{\prod_{n=1}^{m} (s - z_n)}{\prod_{n=1}^{n} (s - p_n)} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + ... + \frac{C_n}{s - p_n}$$

$$\Rightarrow g(t) = \sum C_i e^{p_i t}$$

Note: $C_i e^{p_i t} \rightarrow \infty$ if $\text{Re}[p_i] > 0$

Poles and Zeros

The impulse response of a linear time-invariant (LTI) system

$$Y(s) = \frac{b_n s^n + b_{n-1} s^{n-1} + ... + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0}$$

$$= k \frac{\prod_{n=1}^{m} (s - z_n)}{\prod_{n=1}^{n} (s - p_n)} = \frac{C_1}{s - p_1} + \frac{C_2}{s - p_2} + ... + \frac{C_n}{s - p_n}$$

$$\Rightarrow y(t) = \sum C_i e^{p_i t}$$

($p_i$) are poles of the function and determine the system behavior

Time Response vs. Pole Location

- $f(t) = e^{\sigma t}, p = \sigma + j\omega$

  Undamped Oscillation (critically stable) if $\text{Re} = 0$ and $\text{Im} \neq 0$
  Underdamped Response if $\text{Re} < 0$ and $\text{Im} \neq 0$
  Overdamped Response if $\text{Re} < 0$ and $\text{Im} = 0$
Design Goals
Performance Specifications

- Stability
- Transient response
- Steady-state error

Performance Specifications
Stability

- A LTI system is stable if all poles of its impulse transfer function (i.e., $U(s) = 1$) are in the LHP ($\forall p_i, \text{Re}[p_i] < 0$)

\[
Y(s) = Y(s)U(s) = K \prod_{i=1}^{n} \left( s - p_i \right) = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}
\]

\[\Rightarrow y(t) = \sum C_i e^{p_i t} \]

Note: $C_i e^{p_i t} \rightarrow \infty$ if $\text{Re}[p_i] > 0$

Performance Specifications

- Settling time
- Overshoot
- Control variable
- Time
- Reference
- Steady state error
- Rise time
- Transient state
- Steady state
- Time

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**Example: Control & Response**

- **Good**
  - Control: MaxPackets/Users
  - Response: Queue length
- **Bad**
  - Control: MaxPackets/Users
  - Response: Queue length
- **Slow**
  - Control: MaxPackets/Users
  - Response: Queue length
- **Useless**
  - Control: MaxPackets/Users
  - Response: Queue length

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**Performance Specifications**

**Steady-state error**

- Steady-state (tracking) error of a stable system
  \[ e_{ss} = \lim_{t \to \infty} (y(t) - r(t)) \]

  - \( r(t) \) is the reference input, \( y(t) \) is the system output.
- How accurately can a system achieve the desired state?
- Final value theorem:
  \[ e_{ss} = \lim_{t \to \infty} y(t) = \lim_{s \to 0} sE(s) \]

---

**Performance Specifications**

**Steady-state error**

- **y(t)**
  - \( e_{ss} = -20\% \)
- **R**
  - \( e_{ss} = -20\% \)
- **Time (sec)**
  - \( e_{ss} = -20\% \)

---
### Proportional Control

**Stability of Vegas-like Control**

- **Proportional Controller**
  \[ C(s) = K \]
- **Transfer functions**
  - \( B(s) \) as input: \( T_1(s) = \frac{K}{s+K} \)
  - \( D(s) \) as input: \( T_2(s) = -\frac{1}{s+K} \)
- **Stability**
  - Pole = \(-K\) \(\Rightarrow\) System is stable for \(K > 0\)
- **Steady-state error**
  - For step inputs, steady-state error = \(-D/K\)
  - Steady-state error decreases as \(K\) increases

### Integral Control

**Stability of Vegas-like Control**

- **Integral Controller**
  \[ C(s) = \frac{K}{s} \]
- **Transfer functions**
  - \( B(s) \) as input: \( T_1(s) = \frac{K}{s^2+K} \)
  - \( D(s) \) as input: \( T_2(s) = -\frac{s}{s^2+K} \)
- **Stability**
  - Poles = \(\pm \sqrt{K}\) \(\Rightarrow\) System is critically stable
- **Steady-state error**
  - For step inputs, steady-state error = zero
  - Steady-state oscillation decreases as \(K\) decreases, but rise time increases; tradeoff between transient performance and steady-state performance!