Laplace Transforms

- Formally, Laplace transform for a signal f(t) is:

$$F(s) = \int_0^\infty f(t)e^{-st}dt$$

Complex variable $s = \sigma + j\omega$

- The Power: Ability to study linear systems using algebraic equations
  - Example:

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_1 \dot{u}(t) + b_0 u(t)$$

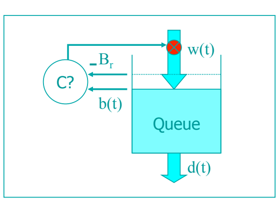
$$\Leftrightarrow Y(s) = \frac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0} \bullet U(s)$$

Laplace Transforms

- Basic translations
  - Impulse signal f(t)=δ(t) ⇔ F(s)=1
  - Step signal f(t)=a.1(t) ⇔ F(s)=a/s
  - Ramp signal f(t)=a•t ⇔ F(s)=a/s^2
  - Exp signal f(t)=e^{at} ⇔ F(s)=1/(s-a)
  - Sinusoid signal f(t)=sin(at) ⇔ F(s)=a/(s^2+a^2)
- Composition rules
  - Linearity L[a f(t)+b g(t)] = a L[f(t)] + b L[g(t)]
  - Differentiation L[df(t)/dt] = sF(s) – f(0) = sF(s) if f(0)=0
  - Integration L[∫f(τ)dτ] = F(s)/s
  - Convolution $y(t) = g(t) * u(t) = \int_0^t g(t-\tau)u(\tau)d\tau \qquad Y(s) = G(s)U(s)$

Vegas-like Model

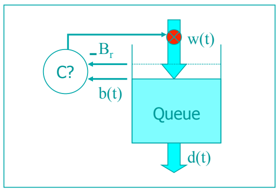
- Error: *e(t)=B$_r$ - b(t)*
- Model (differential equation): $\frac{d}{dt}b(t) = w(t) - d(t)$
- Controller C? *e(t) ⇒ w(t)*



Vegas-like Model

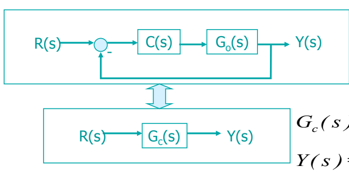
- Error: $e(t)=B_r - b(t)$
- Model (differential equation): $\frac{d}{dt}b(t)=w(t)-d(t)$
- Controller C? $e(t) \Rightarrow w(t)$

$$w(t) = C(B_r - b(t))$$



Block Diagram

- A pictorial tool to represent a system based on transfer functions and signal flows
- Represent a feedback control system



$$G_c(s)=\frac{C(s)G_o(s)}{1+C(s)G_o(s)}$$

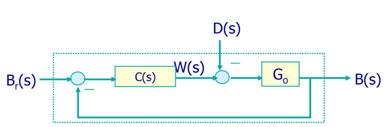
$$Y(s)=G_c(s)R(s)$$

Vegas-like Model
Transfer Function & Block Diagram

- Buffer occupancy is modeled as a differential equation

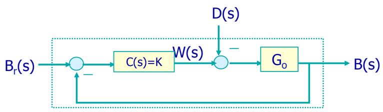
$$\frac{d}{dt}b(t)=w(t)-d(t) \Leftrightarrow B(s)=\frac{W(s)-D(s)}{s} \Leftrightarrow G_o(s)=\frac{1}{s}$$

- Inputs: reference $B_r(s)$; service rate $D(s)$
- Closed-loop system transfer functions
  - $B_r(s)$ as input: $T_1(s) = C(s)G_o(s)/(1+C(s)G_o(s))$
  - $D(s)$ as input: $T_2(s) = -G_o(s)/(1+C(s)G_o(s))$
- Output: $B(s)=T_1(s)B_r(s)+T_2(s)D(s)$



Proportional Control
Stability of Vegas-like Control

- Proportional Controller
  - $w(t) = Ke(t)$; $C(s) = K$
- Transfer functions
  - $B_r(s)$ as input: $T_1(s) = K/(s+K)$
  - $D(s)$ as input: $T_2(s) = -1/(s+K)$
- Stability
  - Pole = $-K$ ⇔ System is stable for K > 0

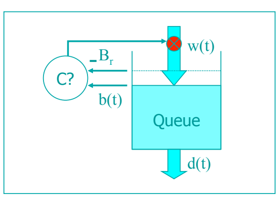


Vegas-like Model (PI controller)

- Error: $e(t)=B_r - b(t)$
- Model (differential equation):
- Controller C? $e(t) \Rightarrow w(t)$

$$\frac{d}{dt}b(t) = w(t) - d(t)$$

$$\frac{d}{dt}w(t) = K(B_r - b(t))$$

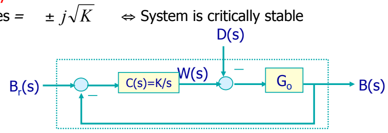


Integral Control
Stability of Vegas-like Control

- Buffer occupancy is modeled as a differential equation

$$\frac{d}{dt}b(t) = w(t) - d(t) \Leftrightarrow B(s) = \frac{W(s) - D(s)}{s} \Leftrightarrow G_o(s) = \frac{1}{s}$$

- Integral Controller
  - $\frac{d}{dt}w(t) = Ke(t)$; $C(s) = K/s$
- Transfer functions
  - $B_r(s)$ as input: $T_1(s) = K/(s^2+K)$
  - $D(s)$ as input: $T_2(s) = -s/(s^2+K)$
- Stability
  - Poles = $\pm j\sqrt{K}$ ⇔ System is critically stable



Performance Specifications Stability

- A linear time-invariant system is stable if all poles of its transfer function are in the left-hand side of the s-plane (∀pi, Re[pi]<0)

$$G(s) = K\frac{\Pi_{i=1}^{m}(s-z_i)}{\Pi_{i=1}^{n}(s-p_i)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

$$\Rightarrow g(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

$$Note: \quad C_i e^{p_i t} \xrightarrow{t\to\infty} \infty \quad if \quad \mathrm{Re}[p_i] > 0$$

Poles and Zeros

- The impulse response of a linear time-invariant (LTI) system

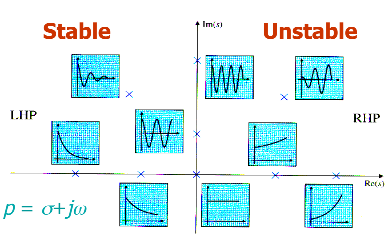
$$Y(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

$$= K\frac{\Pi_{i=1}^{m}(s-z_i)}{\Pi_{i=1}^{n}(s-p_i)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

$$\Rightarrow y(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

{p_i} are *poles* of the function and determine the system behavior

Time Response vs. Pole Location



- *f(t) = e^{pt}, p = σ+jω*

Undamped Oscillation (critically stable) if Re = 0 and Im ≠ 0

Underdamped Response if Re < 0 and Im ≠ 0

Overdamped Response if Re < 0 and Im = 0

Design Goals
Performance Specifications

- Stability
- Transient response
- Steady-state error

Performance Specifications
Stability

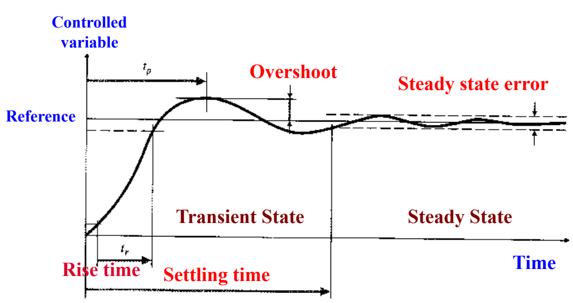
- A LTI system is stable if all poles of its impulse transfer function (i.e., U(s)=1) are in the LHP (∀pi, Re[pi]<0)

$$Y(s) = G(s)U(s) = K\frac{\Pi_{i=1}^{m}(s-z_i)}{\Pi_{i=1}^{n}(s-p_i)} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \ldots + \frac{C_n}{s-p_n}$$

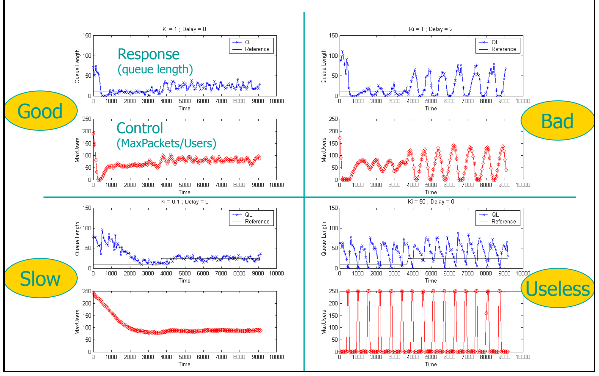
$$\Rightarrow y(t) = \sum_{i=1}^{n} C_i e^{p_i t}$$

$$Note: \quad C_i e^{p_i t} \xrightarrow{t\to\infty} \infty \quad if \quad \mathrm{Re}[p_i] > 0$$

Performance Specifications



Example: Control & Response



Performance Specifications Steady-state error

- Steady-state (tracking) error of a stable system

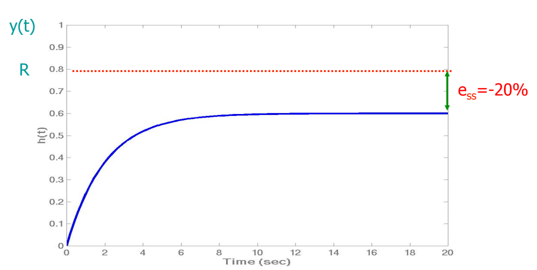
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{t \to \infty} (y(t) - r(t))$$

r(t) is the reference input, *y(t)* is the system output.

- How accurately can a system achieve the desired state?
- Final value theorem:

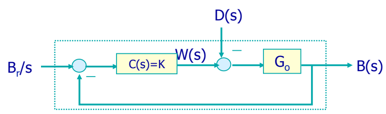
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Performance Specifications Steady-state error



Proportional Control
Stability of Vegas-like Control

- Proportional Controller
  - $w(t) = Ke(t)$; $C(s) = K$
- Transfer functions
  - $B_r(s)$ as input: $T_1(s) = K/(s+K)$
  - $D(s)$ as input: $T_2(s) = -1/(s+K)$
- Stability
  - Pole = $-K$ ⇔ System is stable for K > 0
- Steady-state error
  - For step inputs, steady-state error = – D/K
  - Steady-state error decreases as K increases



Integral Control
Stability of Vegas-like Control

- Integral Controller
  - $\frac{d}{dt}w(t) = Ke(t)$; $C(s) = K/s$
- Transfer functions
  - $B_r(s)$ as input: $T_1(s) = K/(s^2+K)$
  - $D(s)$ as input: $T_2(s) = -s/(s^2+K)$
- Stability
  - Poles = $\pm j\sqrt{K}$ ⇔ System is critically stable
- Steady-state error
  - For step inputs, steady-state error = zero
  - Steady-state oscillation decreases as K decreases, but also rise time increases ➔ tradeoff between transient performance and steady-state performance!