MA/CS 109 Lecture 14

Review

In class exam Thursday here.

Topics

Idea of "proof" in mathematics.

Things you should know: Vocabulary of networks (node, edge, degree of a node, path, circuit, Euler path, Euler circiut). Theorems on Euler circuits and Euler paths.

Idea of proof (what is a proof in mathematics and how does it differ from how conjectures/theories are established in the sciences.)

Topics

Functions: What is a function. How do function differ, particularly how they grow (logs grow slowest, the linear functions, then powers, then exponential functions.) Powers of 2, log base 2.

Relationships between exponential growth and logs (how to recognize exponential growth with logs).

Topics

Population models: Exponential growth model, exponential growth model with harvesting, logistic model. Relationship between assumptions and equations (how to translate assumptions into equations).

Representation of models (time series graphs and population this year vs. population nest year graphs).

Predictions of models: Growth, decline, equilibrium population.

Topics

Probability: Vocabulary (sample space, outcomes, events, probability function, independent events)

Interpretation of probability values (long-term frequency).

The three Axioms of probability. Theorems from the axioms satisfied by all probability function.

Models: Equally likely outcome model for assigning probability. Other possible models.

Topics

Statistics: Vocabulary (mean, median, standard deviation, population, parameter, sample, statistic, confidence level, confidence interval, margin of error.)

Central Limit Theorem: Distribution of sample proportions.

Using proportion from a sample to estimate proportion for a population. Histogram of sample proportions (for samples of size n). Margin of error at 95% confidence level= $1/\sqrt{n}$, n=sample size, (the only formula you should memorize).

Topics

Use Central Limit Theorem to estimate population proportion from a sample proportion. Quality control, “fairness” applications.

Interpretation of poll results and sample procedures and requirements of the Central Limit Theorem.

Problems

A useful exercise is to ask yourself with each sample/homework problem: What is being tested and can I make up another problem that tests the same thing…sometimes you can guess the exam problems (or at least problems like them).

Probability problems

Suppose you are playing a game with a "spinner" that can land one of four colors, red, blue, yellow, green. How many outcomes are there spinning four times?

4*4*4*4=4^4=2^8= 256

continued

What is the probability of getting the same color on all four spins…

This event has 4 outcomes (red,red,red,red or blue,bleu,blue,blue, …) so the probability of this event is

4/256 = 1/64

continued

What assumption/model did you use when you answered the preceding problem and how could you design the spinner so that it followed this model?

Used the equally likely outcomes model.

Each color covers the same angle or arc around the circle and spinner has equal chance of landing on any direction.

continued

What is the probability of getting different colors on all four spins?

Count the outcomes in this event…

First spin has 4 ways to come out, second spin 3ways,third spin 2 and fourth 1. So there are 4x3x2x1 =24 outcomes.

So the probability is 24/256

continued

Is the event of getting all four spins the same color independent from the event of getting all four spins different colors?

Check if Prob (first event and the second event)

Prob (first event) x Prob (second event).

Prob (first event)=4/256

Prob (second event)=24/256

Prob (outcomes in first and second event)

=Prob (all four the same and all four different)=P (empty set)=0

So is (4/256)x (24/256)=0?   NO so these events are not independent.

continued

(Make up a problem…) The preceding example can be generalized to a Theorem that goes like this.

If you have a finite sample space and are using the Equally likely outcomes model, non-empty events A and B (that is, both A and B contain some outcomes) are guaranteed to be dependent if …(complete the sentence and prove the statement).

Theorem and proof

If you have a finite sample space and are using the Equally likely outcomes model, non-empty events A and B (that is, both A and B contain some outcomes) are guaranteed to be dependent if A and B have no outcomes in common then they are dependent (not independent).

Proof: Know…P(A) is not zero (it contains some outcomes and every outcome has a nonzero probability). And P(B) is not zero for the same reason…

But P(outcomes in A and B) =0 because there are no outcomes in A and B (assumption!!).

So in this case P(A) x P(B) can't be zero, so it can not equal P(Outcomes in A and B).

So A and B are non independent (so they are dependent).

Proof

In the natural and physical sciences, a theory is accepted when it is consistently verified by repeated experiments. How does this criterion differ from what is necessary for acceptance in mathematics? Explain in one or two sentences.

Proofs require an explanation of why the statement is true, not just repeated examples of where it is true.

Not everything is sugar coated… (write what you think, not what you guess I want to hear)

What are the limitations of the requirements for acceptance as you outlined above. (That is, how do the requirements necessary for accepting a statement in mathematics negatively impact its usefulness.) (One or two sentences.)

You must understand completely all the terms and the situation…so you must live in some abstract model universe to do a proof in the sense mathematics.

Sadly the physical world isn't always well modeled.

Related question

Where other are statements “proven” in everyday life…(i.e., where do you hear the word “proof” or “proven”)

Proof beyond reasonable doubt…

Related question…

When a jury is given its instructions by a judge, they are required to return a verdict of guilty if they feel the defendant is "Guilty beyond a reasonable doubt".

a) How does this differ from "Proven guilty" in the sense of mathematics. mathematics there is no doubt…

b) How come the mathematical notion of proof is not used for jury trials? Understand everything about the terms and have a clear model universe…

Models

Suppose two investors are very talented. Investor A starts with 2 dollars and investor B starts with 10 dollars. Both investors double their money every week.

(a) Rewrite the sentence above as two equations. A(N)= A's money after N weeks.

B(N) be B's money afte N weeks, then

$A(N+1)=2A(N)$, $A(0)=2$. $B(N+1)=2B(N)$, $B(0)=10$

(b) How much money will investor A and investor B have at the end of 10 weeks?

$A(10)=2^{10}A(0)=1024 \times 2$

$B(10)=2^{10}B(0)=1024 \times 10$

Models

(c) After 20 weeks, how do the totals of investor A and investor B compare?

$A(20)=2^{20}A(0)=1{,}000{,}000 \times 2$

$B(20)=2^{20}B(0)=1{,}000{,}000 \times 10$

(d) Explain how you can be sure that the situation described above can not go on for an entire year.

Not that much money in the world!!

Models

Suppose an invasive species has population that is well approximated by the exponential growth model with a growth rate constant of 1.4.

(a)Write this as a model equation relating population this year to population next year where P(N) is the population in year N.

$P(N+1)=1.4\ P(N)$

Models

Suppose we decide this species tastes good, so we start to harvest. The harvesting rate is regulated to be 1/2 the population each year.

(a) What is the new model for the population?

$P(N+1)=1.4\,P(N) - (1/2)P(N)=1.4P(N)- 0.5P(N) =0.9P(N)$

(b) Is this type of harvesting “sustainable” over the long-term?

Population decreases with this model approaching zero.

Words and formulas

We spent time translating assumptions in words to formulas that we could use…Here's another type of problem on the same topic.

Let P(N) be the population of species P in year N and Q(N) be the population of species Q in year N. Suppose these two species interact in a way that is grows as the population of either species grows…so like P(N) times Q(N).

Words and formulas

Consider the two models

Model 1:

$P(N+1)=0.9P(N) + 0.2\ Q(N)xP(N)$

$Q(N+1)=1.2Q(N) +0.4\ Q(N)xP(N)$

And Model 2:

$P(N+1)=0.9\ P(N) +0.2\ Q(N)xP(N)$

$Q(N+1) = 1.2Q(N) - 0.4\ Q(N)xP(N)$

One of these models is a predator-prey model (one species eats the other) and the other is "symbiotic" or cooperative. Which model is which?

Model 2 is predator prey—Q prey, P predator..

Predator-Prey

$P(N+1)=0.9\,P(N) + 0.2\,Q(N) \times P(N)$

$Q(N+1) = 1.2Q(N) - 0.4\,Q(N) \times P(N)$

Which species is predator, which is prey?

Q is prey.

Can the predators survive without the prey (i.e., does the model indicate they have enough to eat without the other species?)

No P population decreases when Q is zero…

Statistics

After graduation you start work for a company that makes widgets. Your company gets a contract for 100,000 widgets, but you will only get paid if less than 15% of the widgets are defective. Suppose 1000 widgets are tested and 200 are defective. Is the company that bought the widgets justified if they refuse to pay your company?

Sample has 20% defective…but you say

Margin of error is 1/1000…3%   95% confident we shouldn’t get paid…

Statistics

How many widgets would you need to test if you wanted a margin of error at the 95% confidence interval of the form

p-0.005 to p+0.005 (.005 is ½%)

Where p is the proportion of defective widgets in your sample. (No calculator, so what is the formula for the sample size?)

1/(square root of sample size)=0.005

So sample size would need to be 1/ 0.005^2=40,000

Statistics

A drug company is taken to court because if failed to notify its investors that a test of a new drug it was selling had a dangerous side effect. In its defense, the company said the following:

In the population as a whole, the proportion of people having the side effect is 0.15 (or 15%). In our study of 100 people getting the drug, 24% developed the side effect. The margin of error at the 95% confidence level for a sample of size 100 is 1/10=0.1 or 10%, so the results of the sample are within the 95% confidence intervals of the population as a whole. Therefore, we have no legal responsibility to report this relationship.

What do you think of this argument?

(What would you say if you were the judge?)

Things to remember…

Never trust anyone over 30.

Theorems (Statements with proofs) are true forever.

Exponential growth can't go on forever (and the end comes quickly…)

Probability is quantifying uncertainty.

Statistics means never having to say you are sure.

And…one more question…

Why do I hate “Talk Radio”?