MA/CS 109 Lecture 4

Counting,

Counting Cleverly,

And

Functions

The first thing you learned in “math”

The thing in mathematics that almost everybody is good at is counting…we learn very early how to count.

And we count things…pencils, straws, cards,

When we get older we count mostly one thing—

Money.

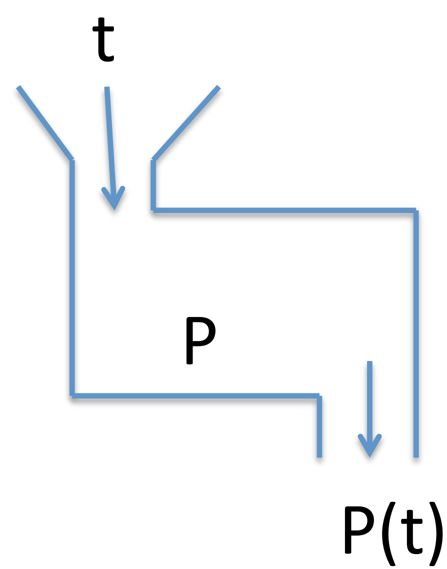
Last Time

Counted the number of ways to assign 50 TA's to 50 classes and to assign 100 TA's to 100 classes (one per class).

But what about 52 TA's to 52 classes or 37 TA's to 37 classes. For each number of TA's and classes there is a number of ways to assign them

Functions

A function is a relationship between two different quantities such that each value of the first quantity is related to one value of the second quantity.



Why are functions nice?

1. They answer a bunch of questions at once (the function giving the number of ways to assign N TA's to N classes tells us how many ways for all possible N at once).
2. We can ask more interesting and sophisticated questions when we have a function—we know a "relationship" between two quantities, not just particular values.

What sort of questions?

How fast does the function value grow as the input grows?

Lots of applications—we know the number of web pages keeps growing every day…it has to take Google longer and longer to search those pages. At what point will there be so many web pages that Google can't possibly search them all? (And will that happen soon?)

Simple example

Here’s another toy example. Suppose a teacher wants to call on a student during a lecture to answer a question.

If there are 5 students, then there are 5 ways to call on a student. If there are 8 students, there are 8 ways to choose the student to call on…if there are 17 students…you get the idea.

IF there are N students, then there are N ways to pick one student.

Notation

Let N be the number of students.

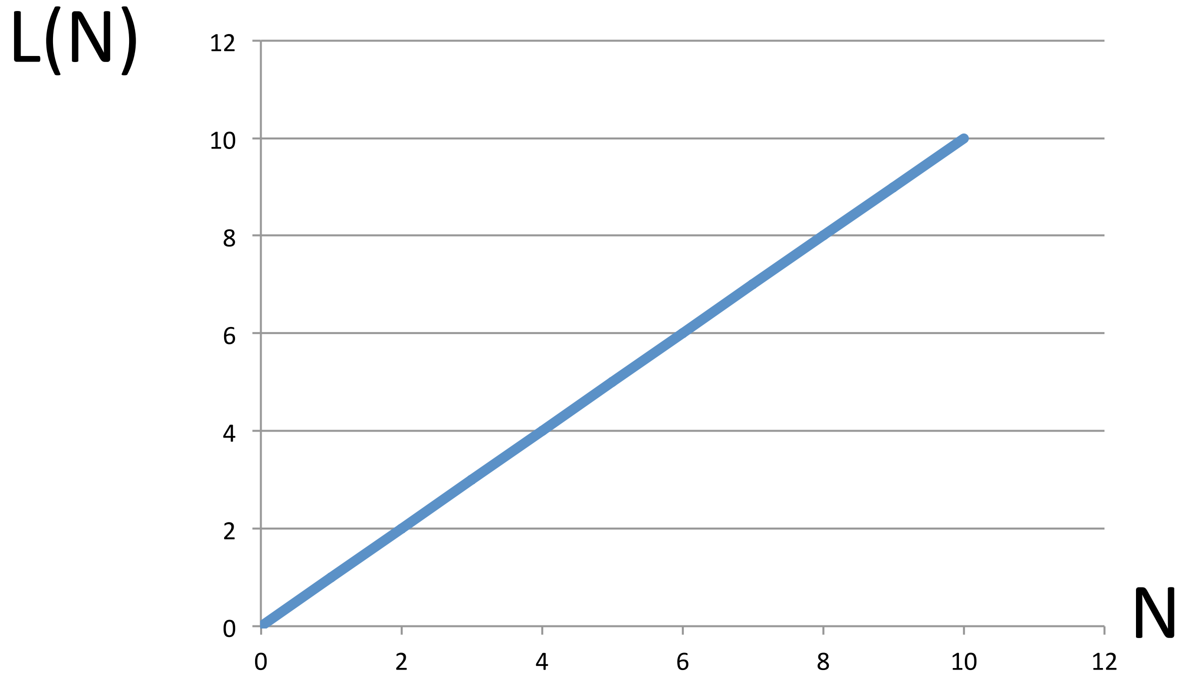
Let L(N) be the number of ways to pick one student to answer a question.

Then L(N)= N (the number of ways to pick a student is the same as the number of students.)

(Why is the function called "L"?)

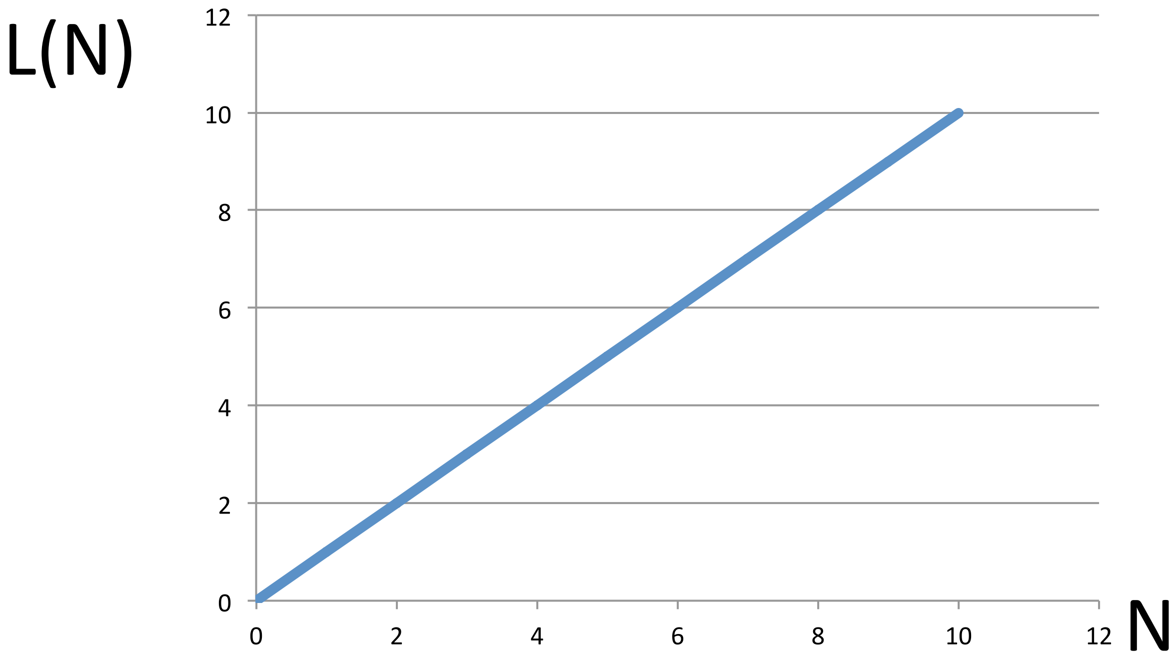
Pay off of abstract view

We can "see" the function—draw a graph. This allows us (among other things) to see how fast it grows as N gets big.



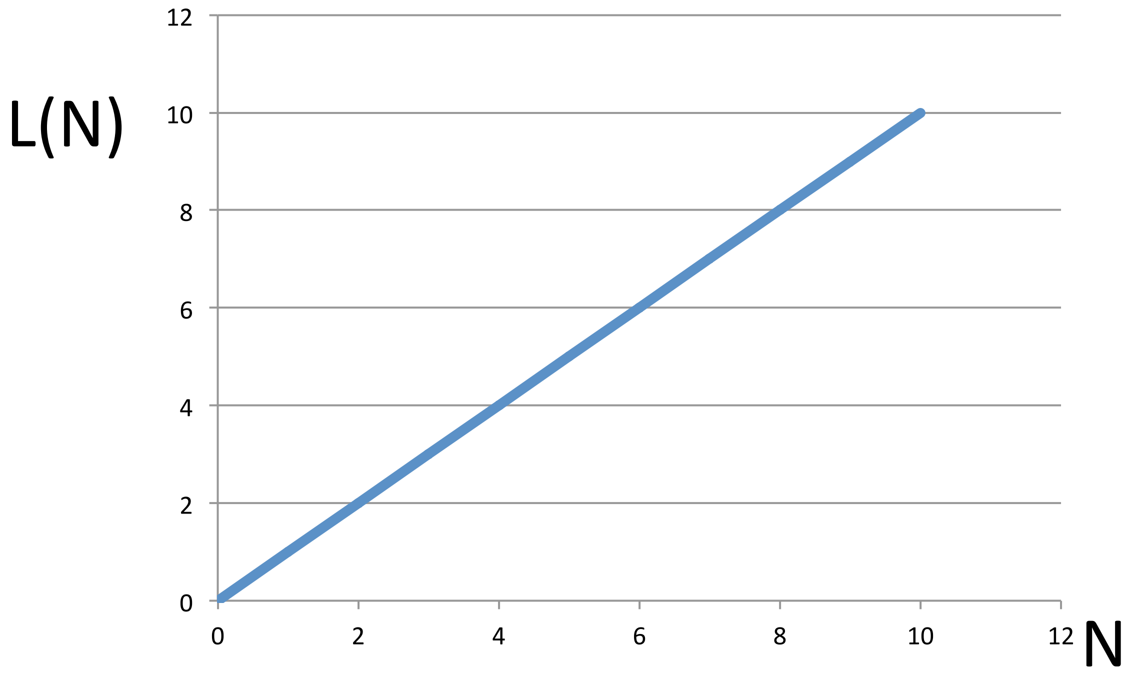
Pay off of abstract view

Notice that the number of students and the number of ways of picking one student grow at the same rate.



Pay off of abstract view

That is the rate of growth of L (the size of L(N+1)-L(N)=(N+1)-N=1 is always the same…Property of lines).



Another Function

Suppose you have N students and you want to pick 2 students to answer questions (you don't care if you ask the same student twice). How many ways are there to choose a student for question 1 and a student for question 2?

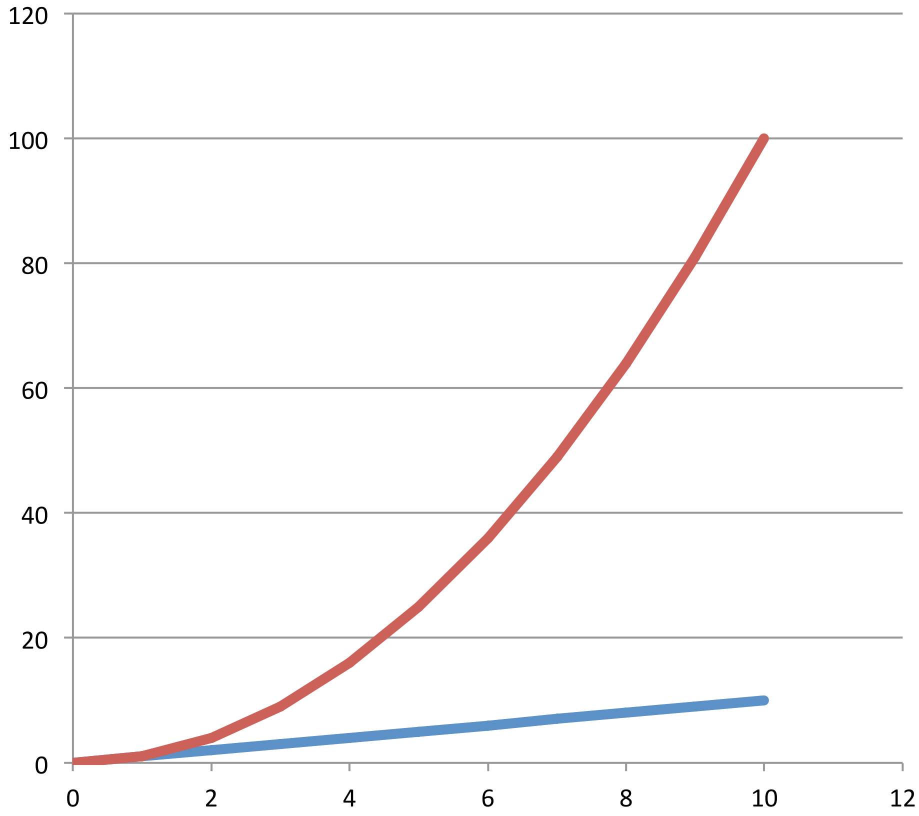
Multiplication Principle again…N ways to pick student 1 and N ways to pick student 2 (CAN pick the same student twice) so

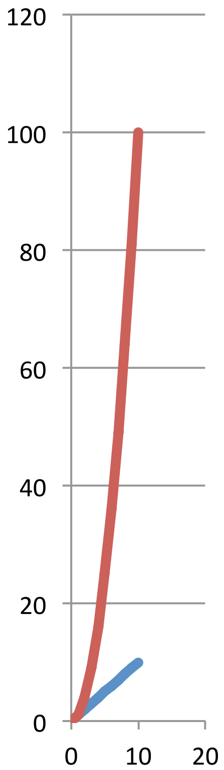
$$Q(N) = N \times N = N^2$$

ways. (Why "Q"?)

How fast does Q grow?

Red is Q, blue is L…(Note scale)





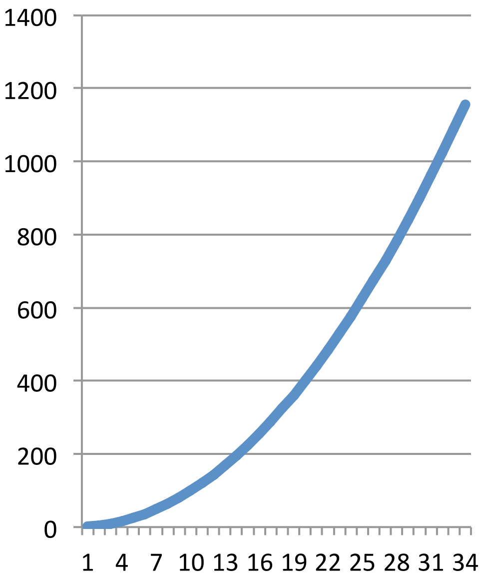
Keep going.

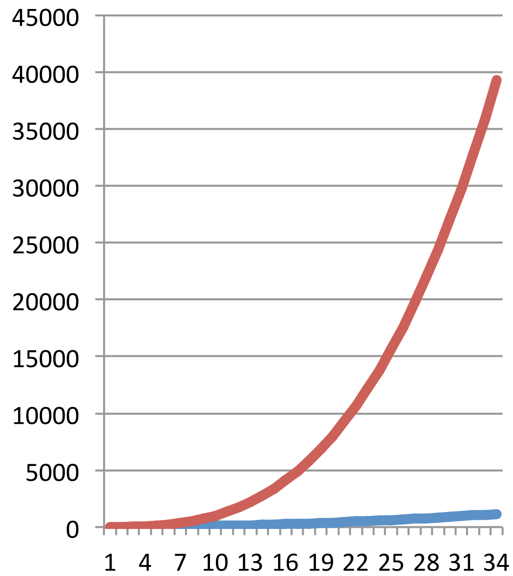
Choose 3 students from N (repeats allowed) for questions 1, 2 and 3: $C(N)=NxNxN=N^3$

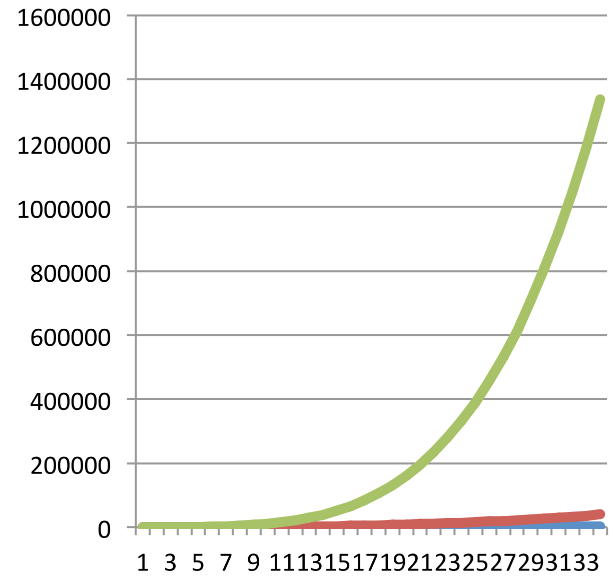
Choose 4 students from N (repeats allowed) for questions 1,2,3 and 4: $F(N) = NxNxNxN = N^4$

Keep going all day…

The bigger the power, the faster the growth…







Blue is quadratic growth ($Q(N)=N^2$) in all three graphs,

Red is cubic growth ($C(N)=N^3$) in middle, right graphs,

Green is quartic growth ($F(N)=N^4$) in the right graph.

“Rate of Growth”

We say that the “rate of growth” of the cubic function is faster than the quadratic function because eventually (for N big) N^3 is hugely bigger than N^2.   The rate of growth of the fourth degree quartic function is faster than the cubic function…and so on…

Rate of Growth: A little algebra

How fast does $N^2=Q(N)$ grow?

How much does Q change when N is changed by 1.

Well, look at how much $Q(N)=N^2$ changes from N to (N+1). We compute

$$(N+1)^2 - N^2 = N^2 + 2N + 1 – N^2 = 2N +1$$

Increasing N by 1 increases N^2 to $(N+1)^2$, the amount of increase is by 2N+1.

So the rate of growth of N^2 is proportional to N.

(The larger N is the faster N^2 grows.)

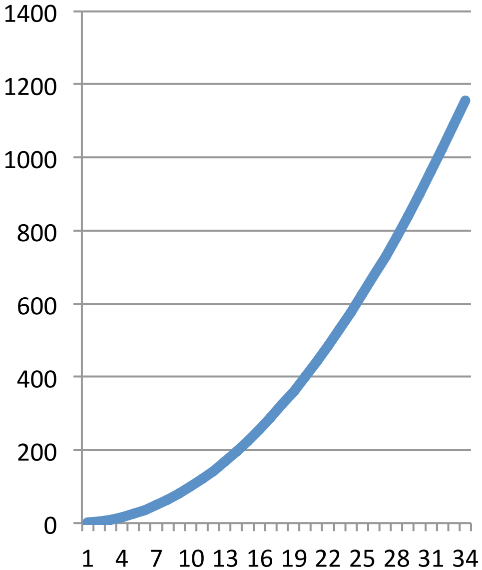
“Rate of Growth”

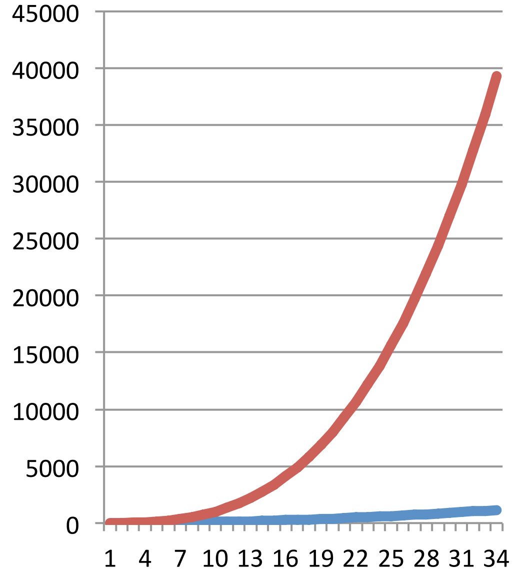
Turns out $C(N)=N^3$ grows at a rate that increases quadratically (like $Q(N)=N^{2)}$,

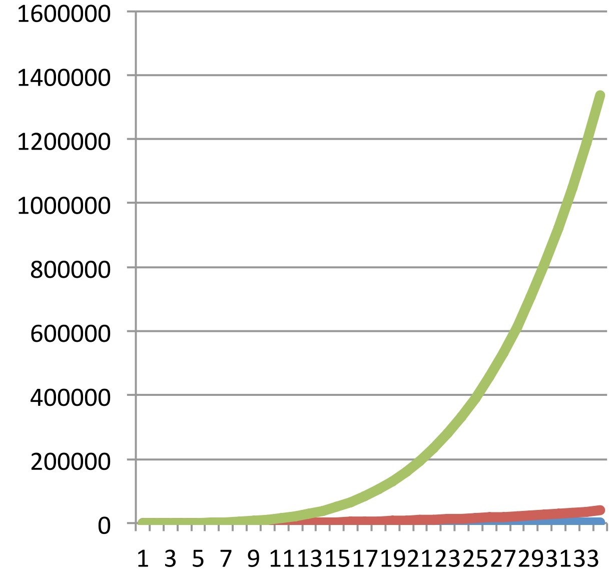
While $F(N)=N^4$ grows at a rate that increases cubically…and so on.

So—Moral—The bigger the power the faster the growth—Lots and lots faster!

Can’t even see quadratic function grow on the right…







Blue is quadratic growth (N^2) in all three graphs,
Red is cubic growth (N^3) in middle and right graph,
Green is quartic growth (N^4) in the right graph.

Even Faster Growth?

Are there functions that grow faster than any power?…and do these functions occur in our daily lives??

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YES! Exponential Growth!

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YES! Exponential Growth!

Exponential growth have a rate of growth related to the size of the function—So…

The Bigger the value, the Faster it Grows…

The Bigger it is the Faster it Grows

Sounds like the tag line of a horror movie…

And sometimes it is…you have a dangerous generator of exponential growth lurking in your wallet right now…

(Ba Dah, Ba Dah, Ba Dah,…..)

CREDIT CARD DEBT!!!

Any sort of compound interest—that is any time when you hear

“The amount you owe grows by ....% per year”

This is saying the amount you owe grows at a rate proportional to what you owe—so the more you owe, the faster your debt grows—

EXPONENTIAL GROWTH!!!

How fast is Exponential Growth?

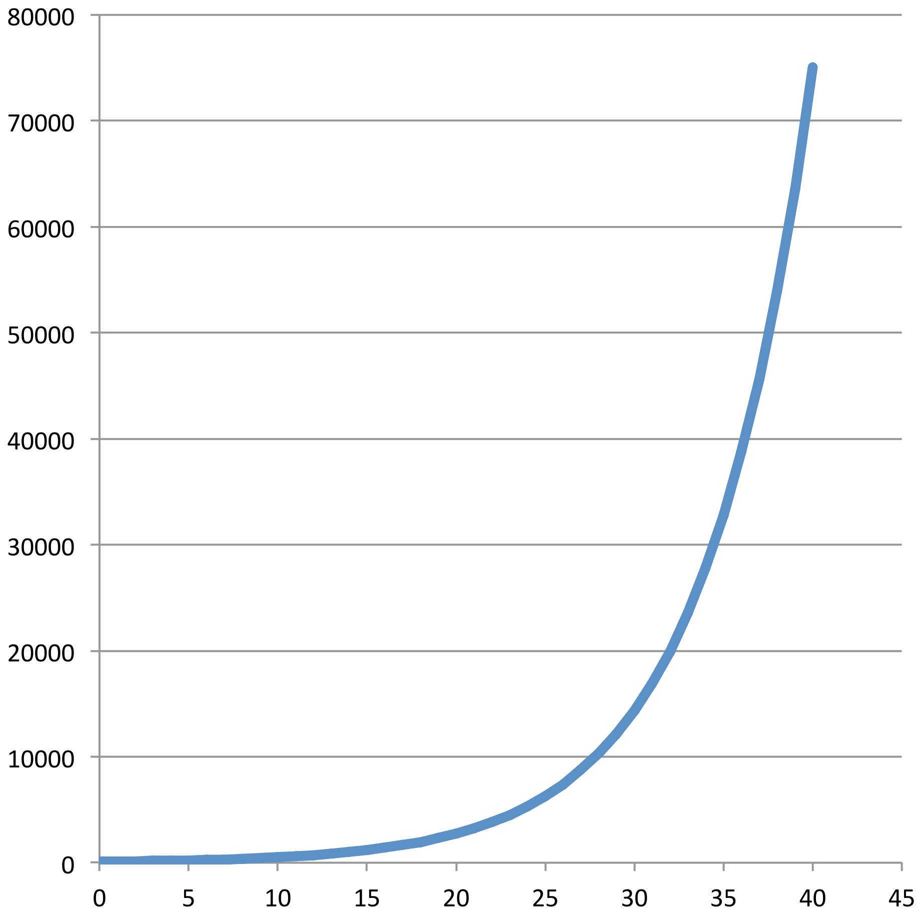
Suppose you charge $100 on a credit card you get for free in college…you forget the charge and move away and the credit card company doesn't find you until your 40^{th} reunion…

How much do you owe now (assuming 18% per year interest).

First year 100 + 0.18 x 100 = 118

Second year 118 + 0.18 x 118 = 139.24

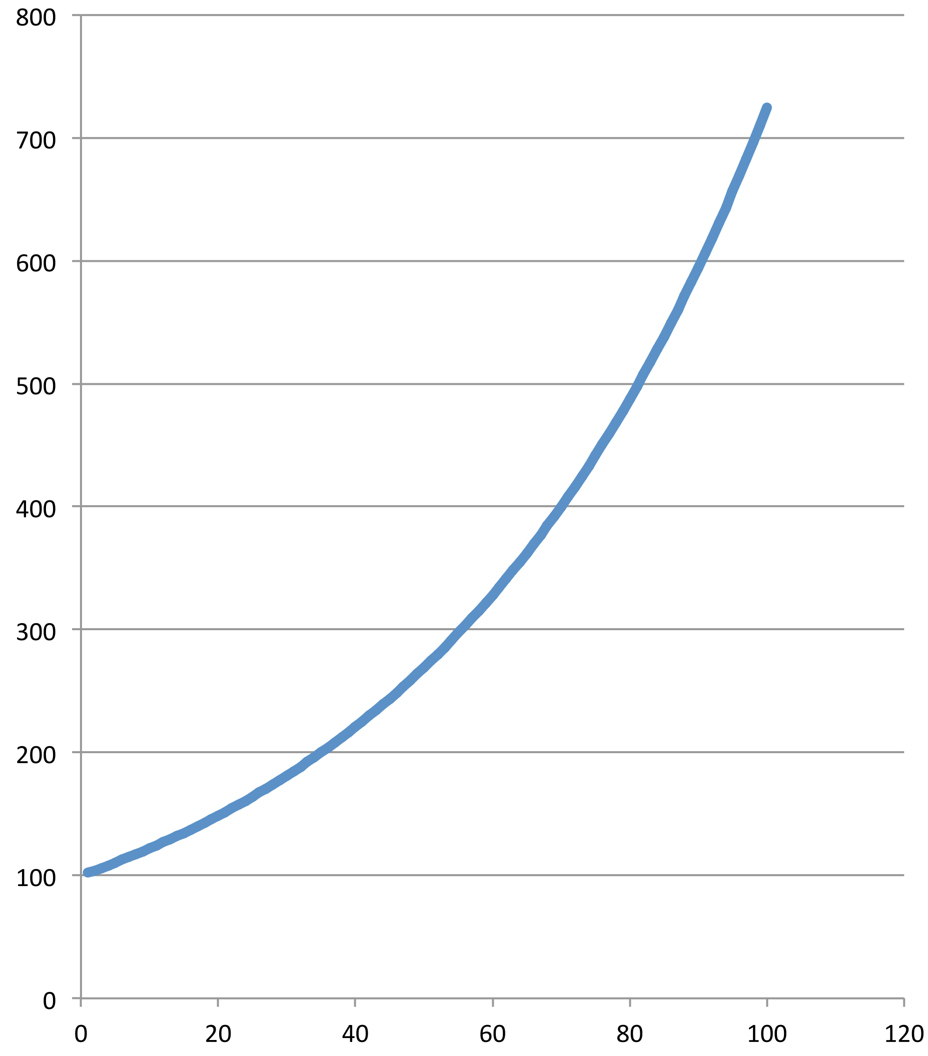
After 40 years…debt= $75,037.83



Characteristic shape of graphs of exponential growth.

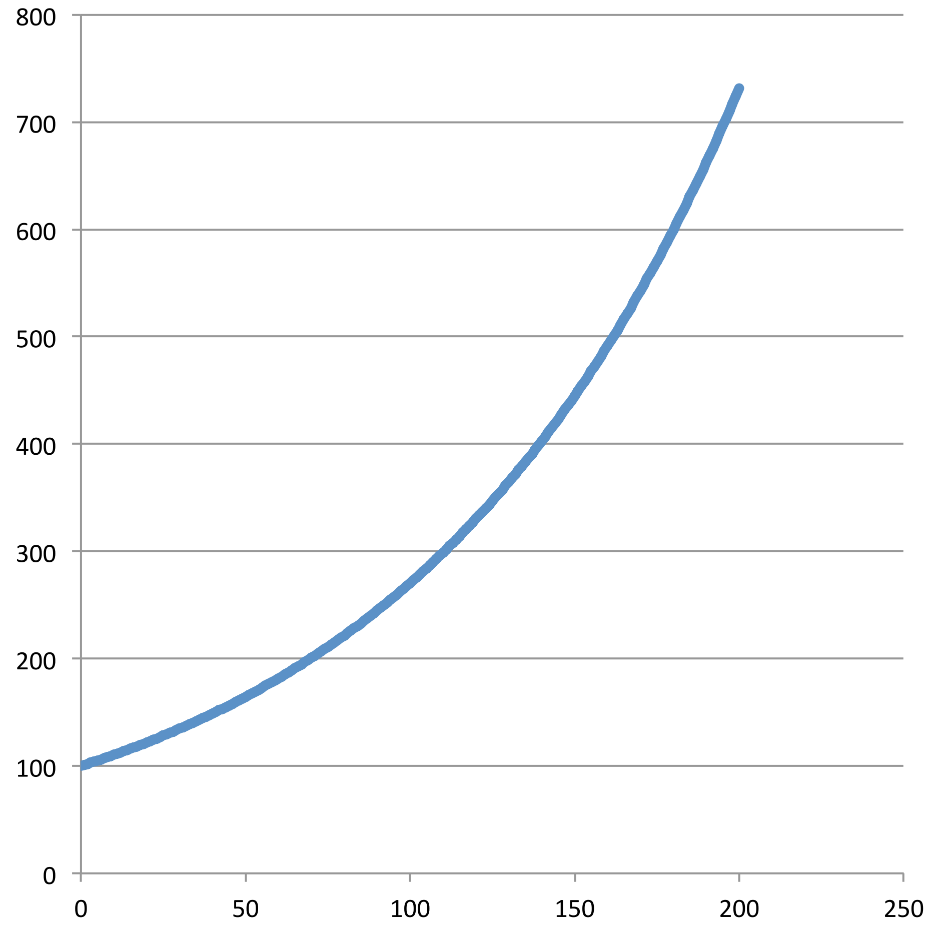
Even small percentage rates have this shape.

CO_2 in the atmosphere grows at a rate of 2% per year… (increases by 600 percent in 100 years…)



Even small percentage rates have this shape.

CO_2 in the atmosphere grows at a rate Of 1% per year… (increases by 170 percent in 100 years, 630 percent in 200 years)



The most important things you will learn in college…or ever…

The most important things you will learn in college…or ever…

1. Never trust anybody over 30.

2. Exponential growth can not go on forever.

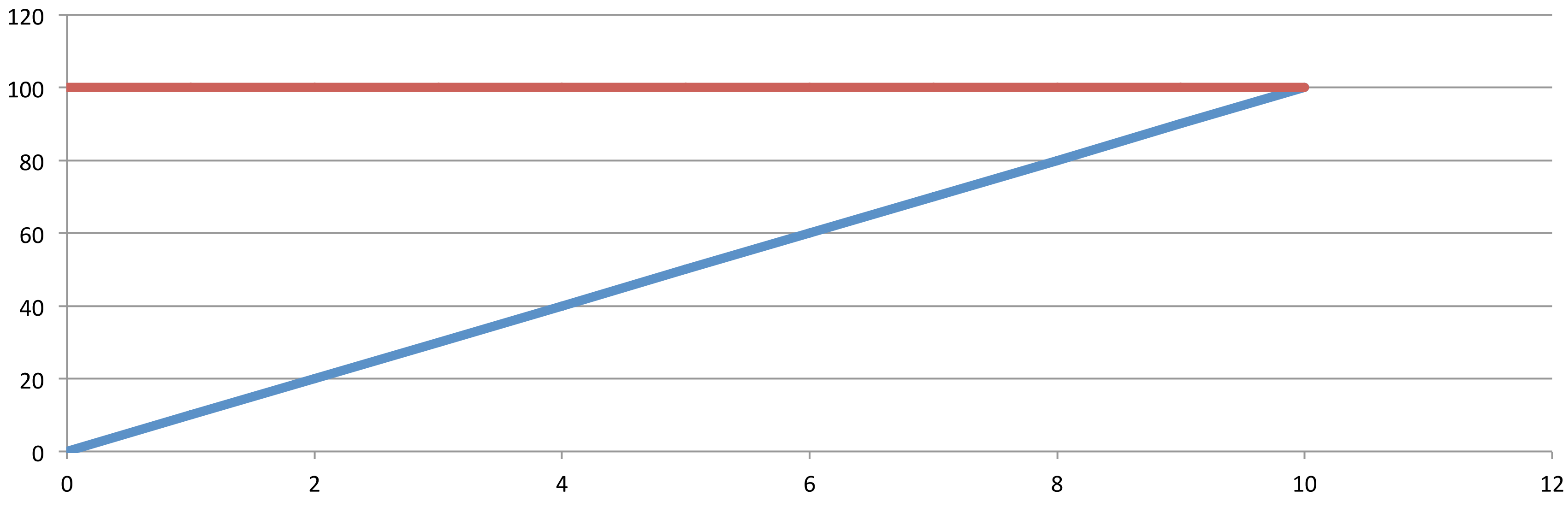
D’uh…

Of course exponential growth can’t go on forever. Quadratic, even linear growth can’t go on forever on a finite planet…

But exponential growth is a lot more devious and potentially dangerous because of the shape of the graph…

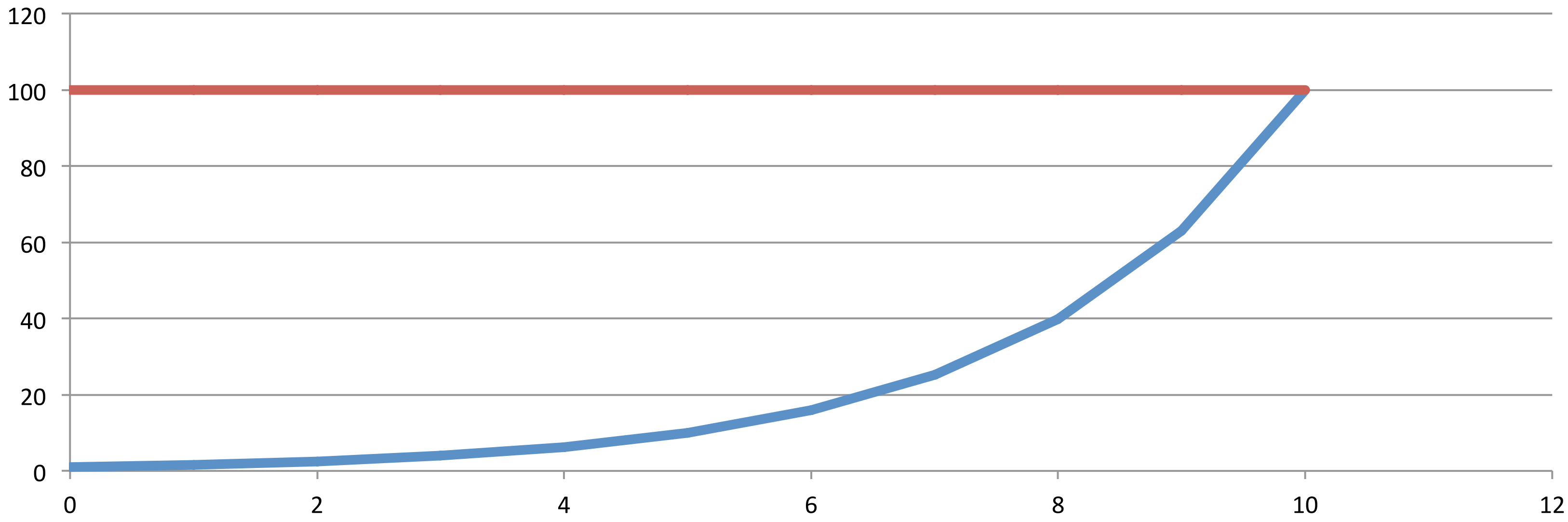
Linear growth to 100

Note that for linear growth, if we reach the limit at time 10, we are half way to 100 at time 5 and nine tenths of the way to 100 at time 9…We see the end coming.



Exponential growth to 100

For exponential growth, if we reach the limit at time 10 then at time 5 we are only at about 10% of the way to 100, at time 9 we are still only at 63%. The fastest growth comes at the end. The limit “sneaks up on us”.



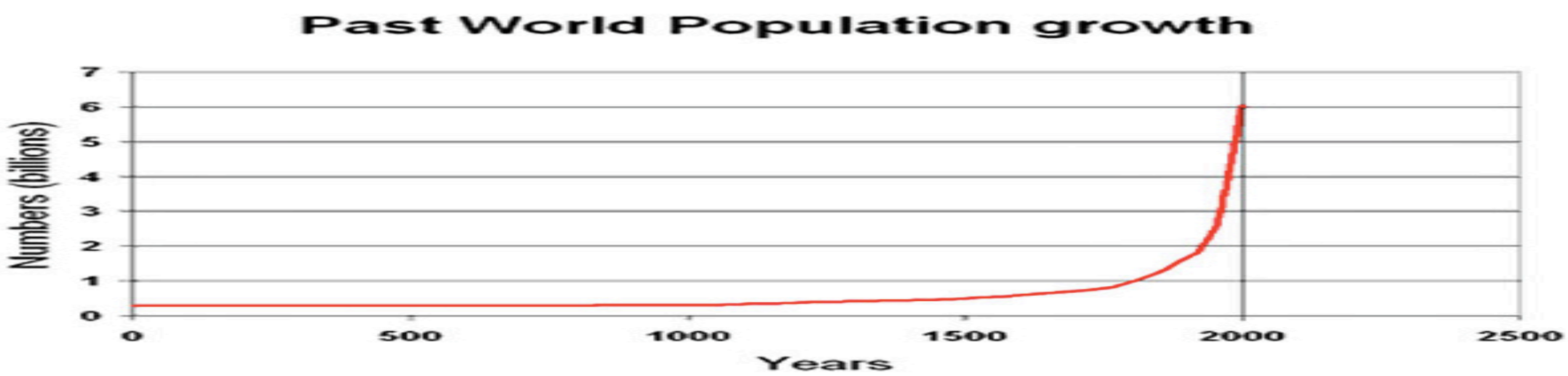
This is what makes exponential growth dangerous. The rate of increase increases so fast that the values eventually grow very quickly. Everything seems find and suddenly---BAMM, you are at the limit.

This doesn't mean the world ends or there is catastrophe…at least it doesn't ***always*** mean that.   But it does mean that things will change and if the limit of growth is strict then change will come quickly!

Example…

World population growth has been exponential for millennia…and predictions of dire catastrophe because of overpopulation have been around for a long time…

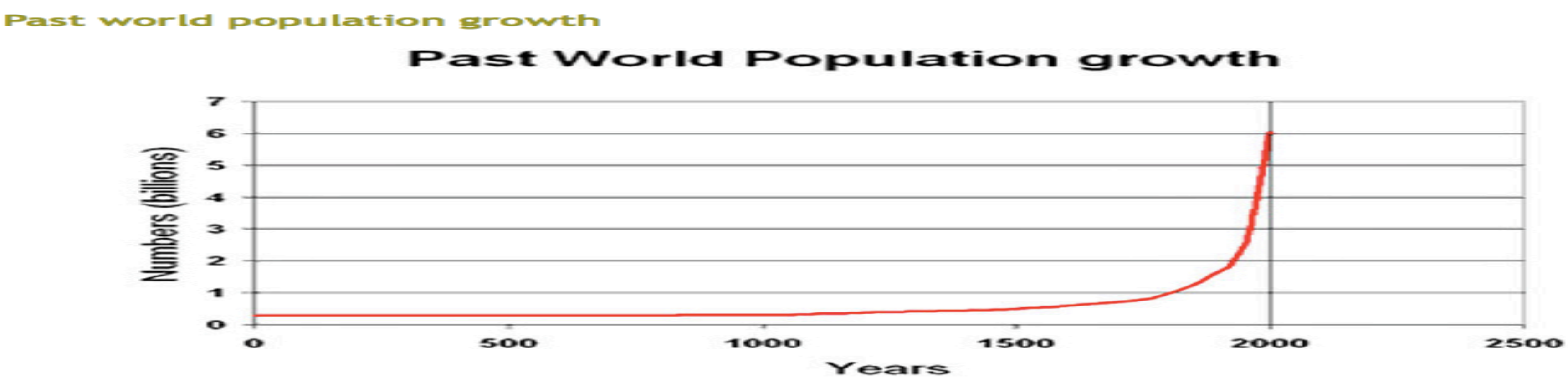
Past world population growth



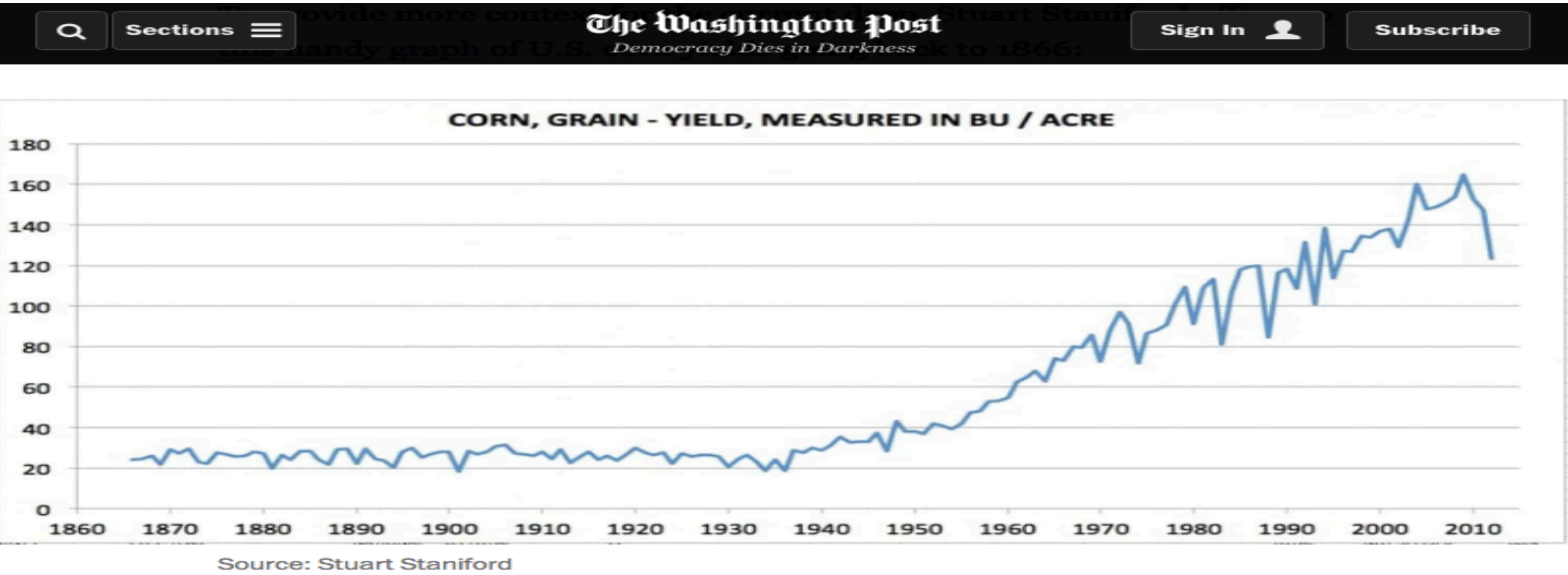
From https://www.learner.org/courses/envsci/visual/visual.php?shortname=past_pop_growth

Example…

Predicting the “end of the world as we know it” because of population growth is actually pretty accurate…For example, look at agriculture



Yes, we would have run out of food due to population growth—except that something changed! We (as a species) got a lot better at growing food! Also, cultural change has reduced population growth in many countries



Glass half full!

The above is the pessimist's view…exponential growth is dangerous!

The optimist says "Thank goodness for exponential growth. It is a miracle." By investing in an account that has exponential growth (where the return is a percent of the investment) my money will eventually grow very fast…I will retire rich!!

So if you can recognize exponential growth, then you can expect there to be change (good or bad)—and the change may have to happen rapidly.

If you see change coming…it is like seeing the future!

Two problems:

1. How do we recognize exponential growth.
2. How do we predict <u>what</u> changes will happen?

Easier problem…

1. How do we recognize exponential growth…

Need another new function. Turns out this function is extremely useful…but also you need to think to understand it.

Logarithms!

https://opinionator.blogs.nytimes.com/2010/03/28/power-tools/

(By the way, this is a nice article…)

Logarithms

We use the type of Logs that computer scientists love…Logs base 2:

Log base 2 of a number a is the number b such that 2^b=a. We write $\text{Log}_2(a) = b$.

So $\text{Log}_2(2) = 1$ because $2^1=2$

$\text{Log}_2(4) = 2$ because $2^2=4$

$\text{Log}_2(8) = 3$ because $2^3=8$

Quiz

$Log_2(16) = 4$ because $2^4=16$

$Log_2(32) = 5$ because $2^5=32$

Quiz

$Log_2(16) =$ because $2^?=16$

$Log_2(32) =$ because $2^?=32$

$Log_2(25) =$ because $2^?=25$

Quiz

$Log_2(16) = 4$ because $2^4=16$

$Log_2(32) = 5$ because $2^5=32$

$Log_2(25) =$ between 4 and 5 because $2^4=16$ and $2^5=32$.

(Ask Google…

$Log_2(25) \sim 4.644\ldots$ because $2^{4.644}$ is about 25.

Impress Prof Snyder…

Computer Scientists use powers of 2 (and Log_2) a lot because they use the TWO digits 0 and 1 to represent states of a "bit"…

To impress Prof Snyder (and make the CS part of the class easier…Memorize

$2^1=2$	$2^6=64$
$2^2=4$	$2^7=128$
$2^3=8$	$2^8=256$
$2^4=16$	$2^9=512$
$2^5=32$	$2^{10}=1024$

And Remember

2^{10} = 1024 which is approximately 1000

(so $Log_2(1000)$ = 9.966… which is almost 10

Note that Log_2 of 1000 is only (about) 10…

$Log_2(2000)$ is about 11,
$Log_2(4000)$ is about 12…
So Log_2 grows really, really slowly

Use property of Log_2

In fact, Log_2 grows so slowly that it turns "exponential growth" (very fast growth) into "linear growth".

(This is the property of Logs that goes

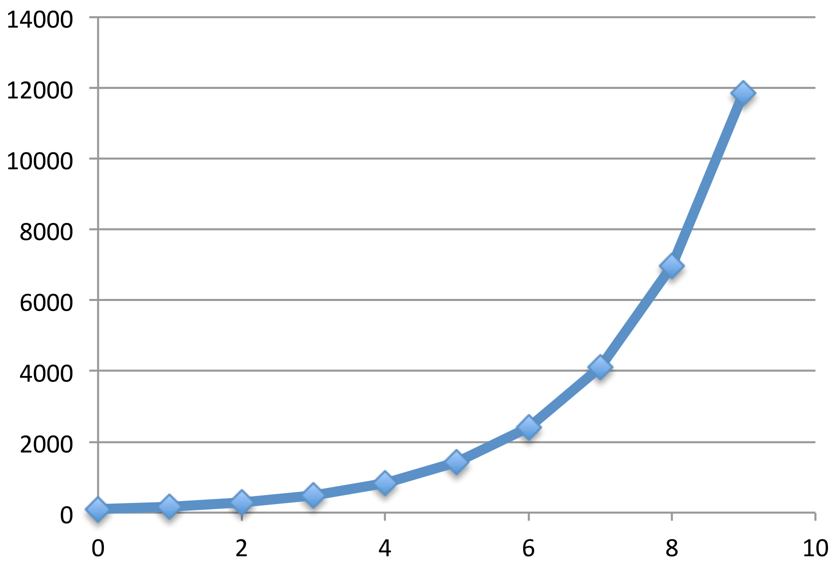
$$Log_2(a^N) = N(Log_2(a)).$$

Logs turn exponentiation into multiplication…)

Looking at Data

So suppose I have some data:

Time	Money
Time 0	100
Time 1	170
Time 2	289
Time 3	491
Time 4	835
Time 5	1491
Time 6	2413
Time 7	4103
Time 8	6975
Time 9	11858



Displays rapid growth

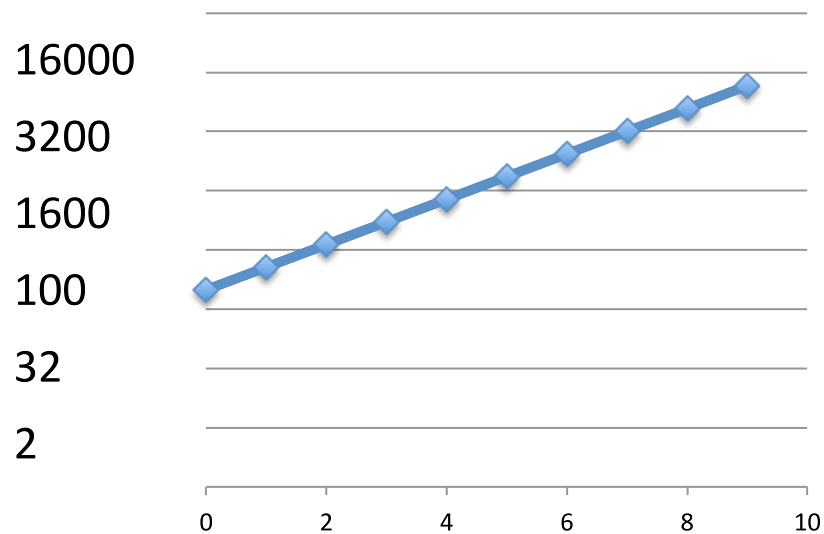
Is it Exponential growth…?

Log charts

Compute the Log_2 of money and plot that

That is, make a "Log_2 plot"--

Time	Money	Log base 2 (money)
0	100	6.643
1	170	7.409
2	289	8.175
3	491	8.939
4	835	9.706
5	1491	10.471
6	2413	11.237
7	4103	12.002
8	6975	12.768
9	11858	13.533

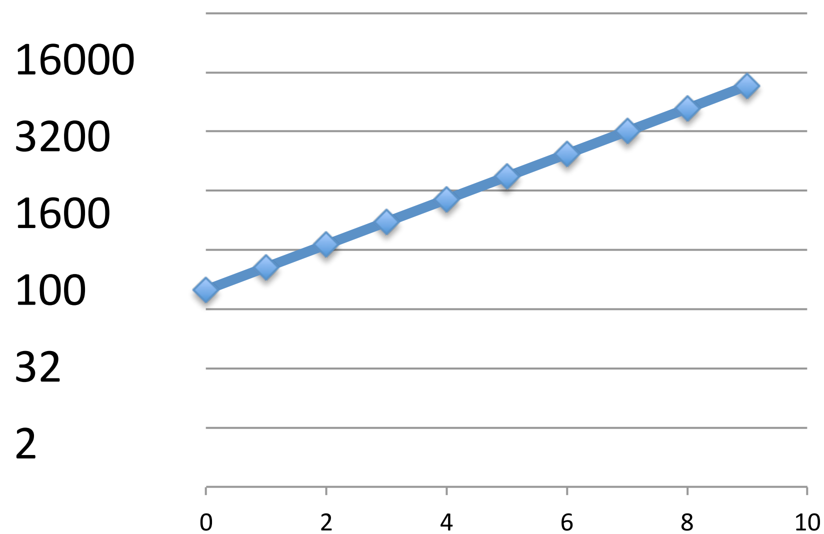


The Log plot is a line!

Log charts

You have a “Log chart” if the vertical axis is not “evenly” divided…more spread out.

Time	Money	Log base 2 (money)
0	100	6.643
1	170	7.409
2	289	8.175
3	491	8.939
4	835	9.706
5	1491	10.471
6	2413	11.237
7	4103	12.002
8	6975	12.768
9	11858	13.533



The Log plot is a line!

One more example of exponential growth…

Croissant. Make a croissant by encasing a layer of butter between 2 layers of dough. Then you spread (roll out) this package into a thin sheet and fold so that there are now three layers of butter.

You repeat the process and 3 layers become 9,

Repeat again and 9 layers become 27…

Each repeat is called a “turn” (because you rotate the dough 90 degrees before folding).

How many layers after 4 turns?

Why not do 10 turns?...