# MA/CS 109 Lecture 5

Logs

And

Exponentials

# Notice

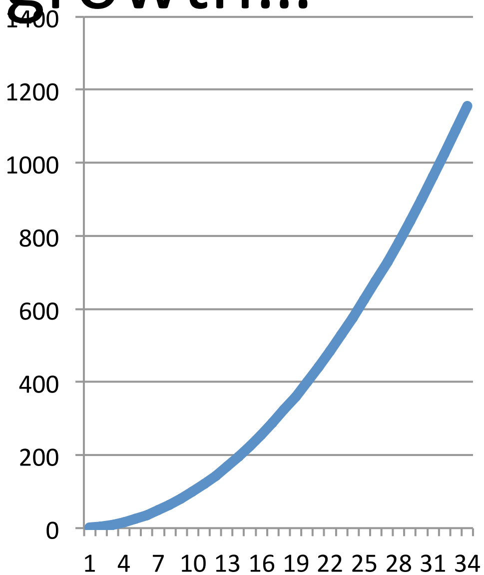
I will not be in the office during tomorrow's office hour.....

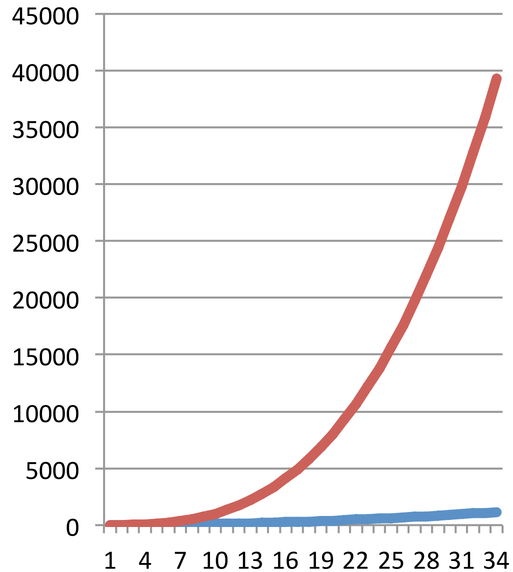
# Last time we went to the Function Zoo

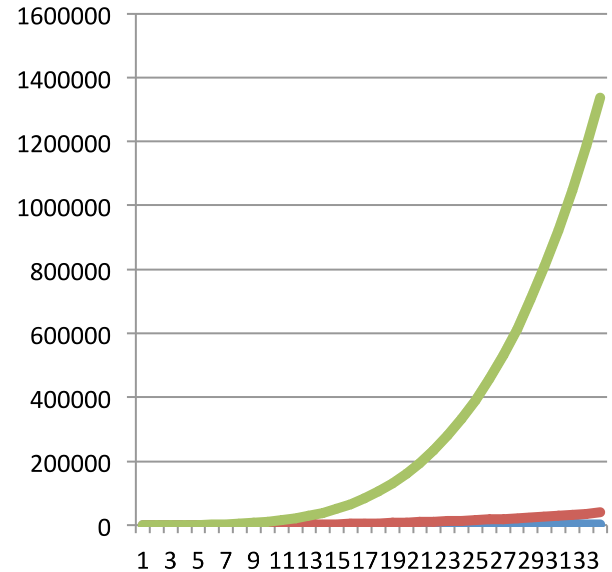
Powers, exponentials, and Log base 2.

In particular, we worried about how fast these functions “grow”, that is, how big value gets when the input variable gets big…

# We saw--the bigger the power, the faster the growth…



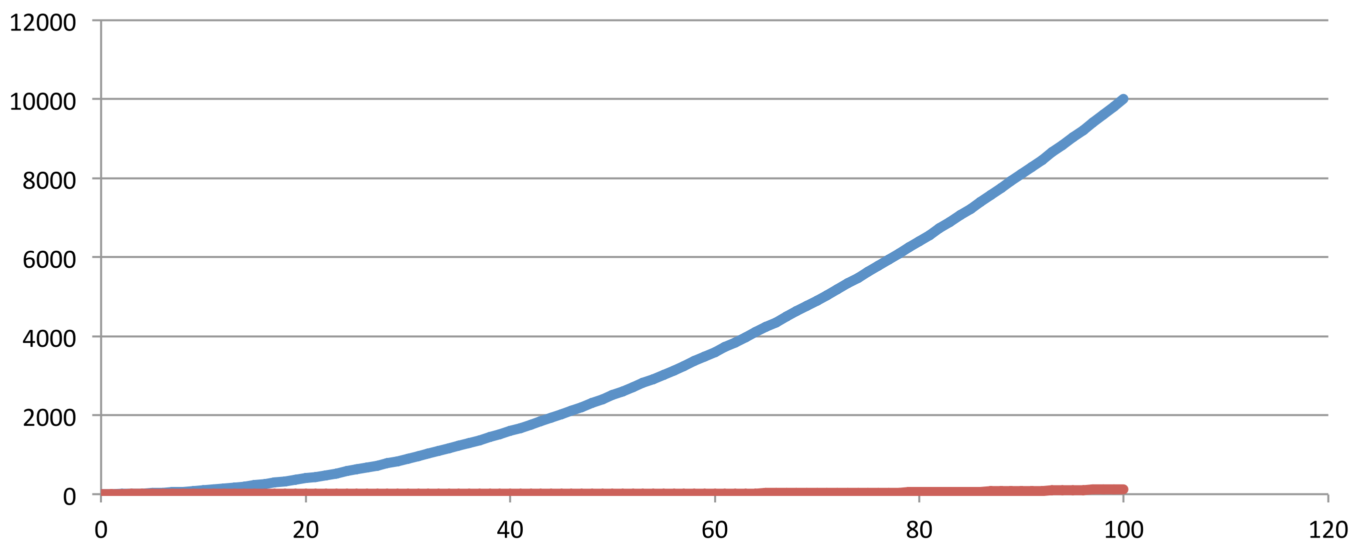

Blue is quadratic growth ($Q(N)=N^2$) in all three graphs,

Red is cubic growth ($C(N)=N^3$) in middle, right graphs,

Green is quartic growth ($F(N)=N^4$) in the right graph.

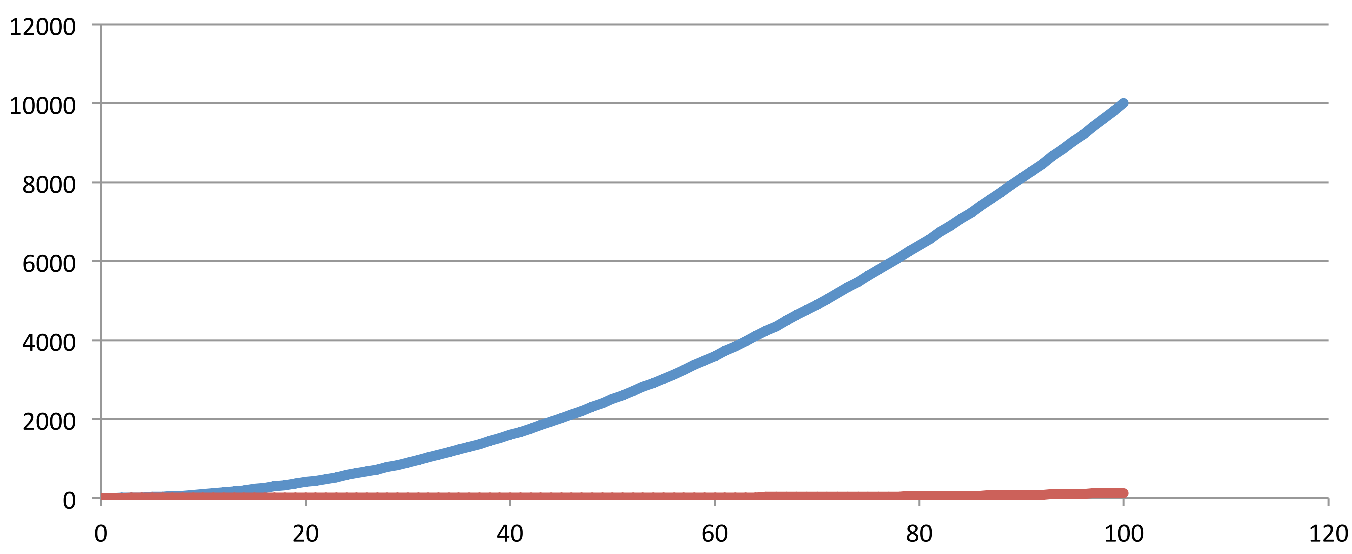
# Exponential Growth faster than any power!

To demonstrate, here's a graph of the quadratic function $Q(N)=N^2$ in year N and the exponential growth, starting at 1, of 5% per year . (Quadratic is blue, exponential is red.)



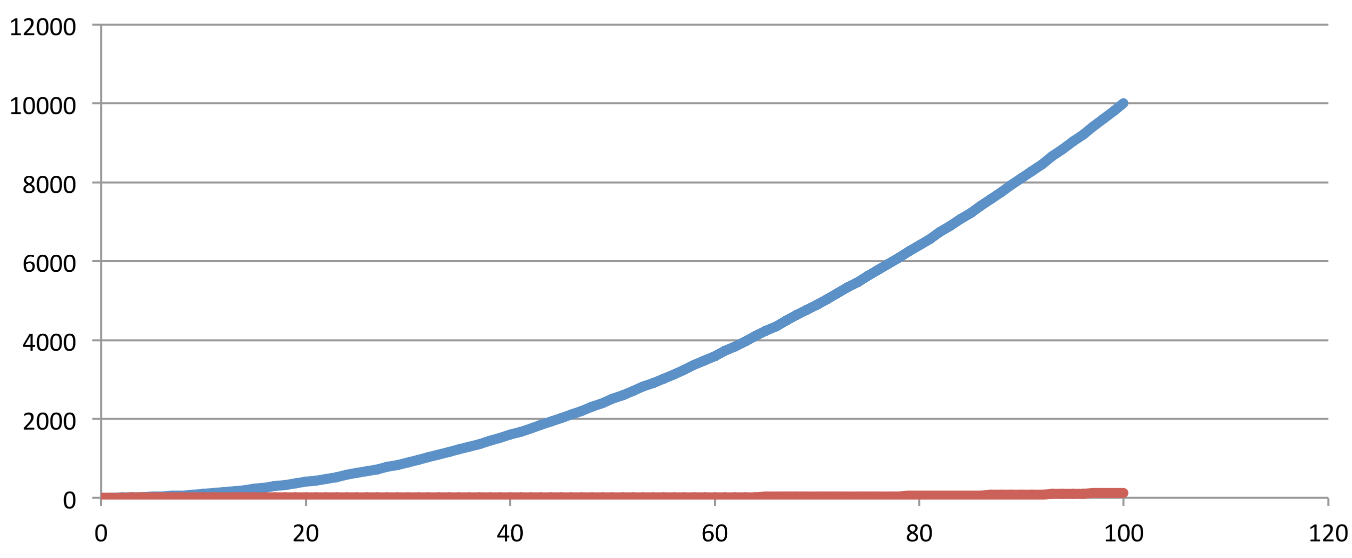
# Exponential Growth faster than any power!

That's embarrassing…the quadratic function is much bigger than the exponential function at 100…



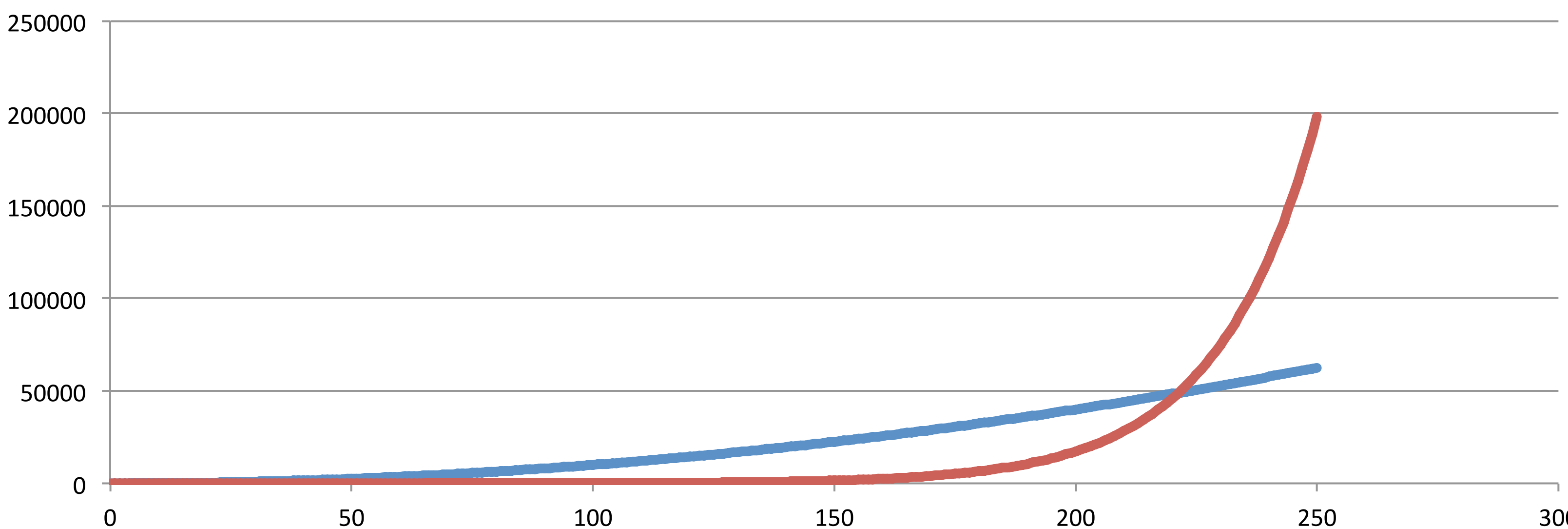
# Exponential Growth faster than any power!

But remember, exponential growth is growth proportional to size—so it won't grow really fast until it is big…lets go farther into the future.



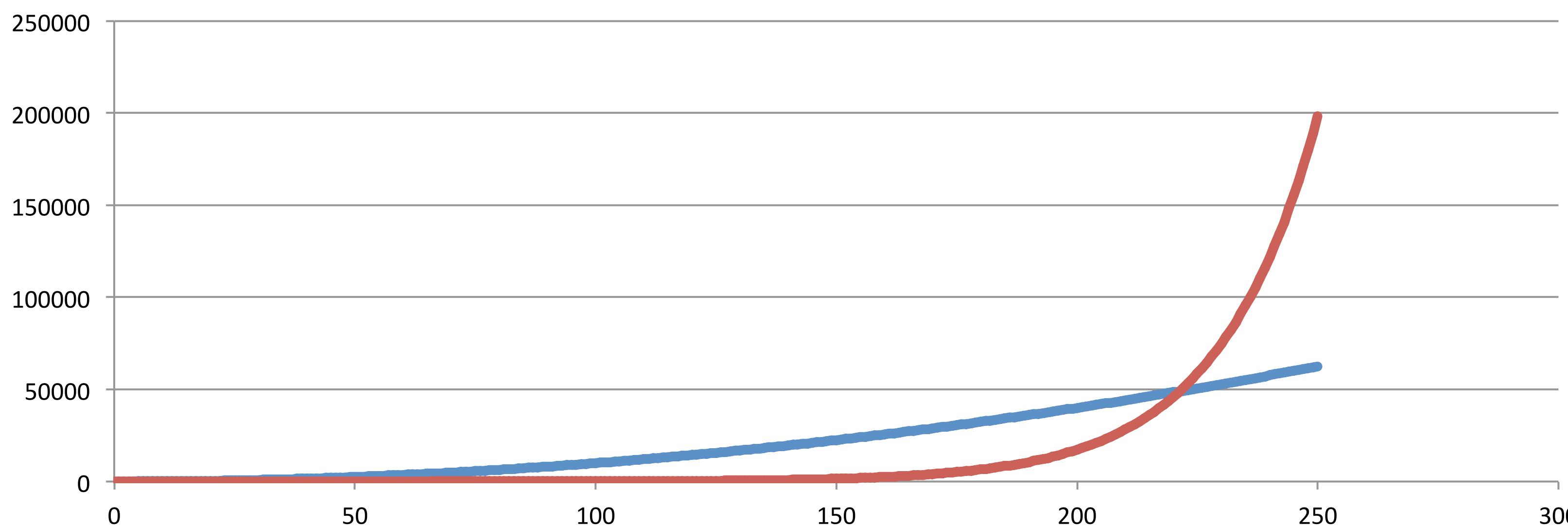
# Exponential Growth faster than any power!

Blue is quadratic and red is exponential—even out to N=200, quadratic is bigger…but then exponential growth (even at just 5%) “takes off” in the “exponential explosion”.



# Exponential Growth faster than any power!

Saying one function grows faster (e.g., that exponential growth is faster than quadratic growth) means one function "eventually is much larger".



# Log base 2:

Our last function was $Log_2$ or “Log base 2”.

$Log_2(a)$ is the number b such that $2^b$= a.

So (quiz)

$Log_2(8) = 3$

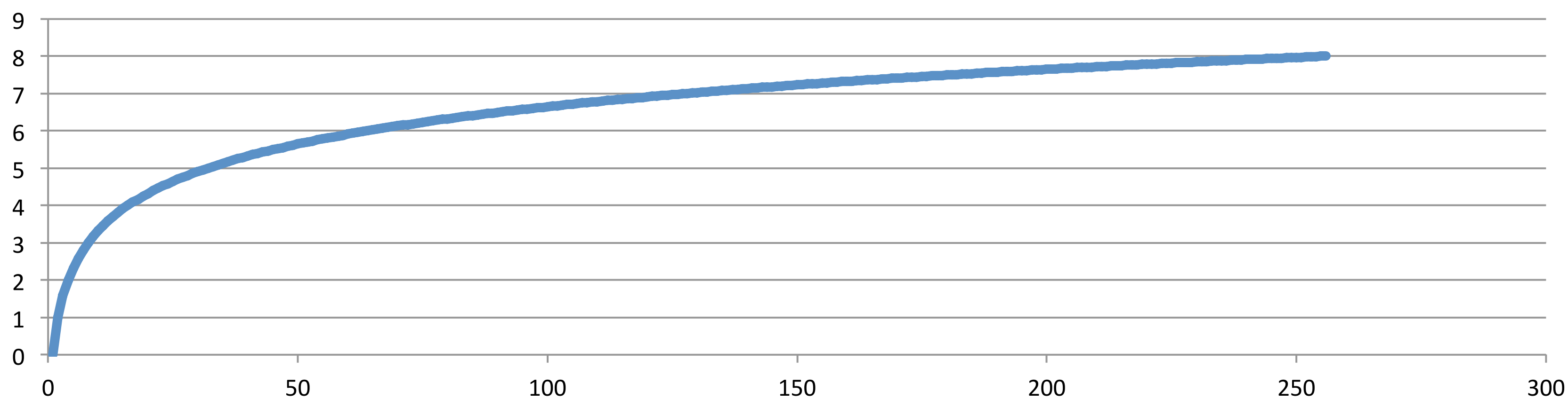
$Log_2( 256) = 8$

$Log_2(1000)$ =abour 10 because $2^{10}$=1024~1000

# $Log_2$ grows really slowly…

Note the scale!! And as N gets bigger, $Log_2(N)$ grows more and more slowly.

In fact, $Log_2$ grows so slowly that it turns exponential growth into linear growth.



# $Log_2$ turns exponential growth into linear growth

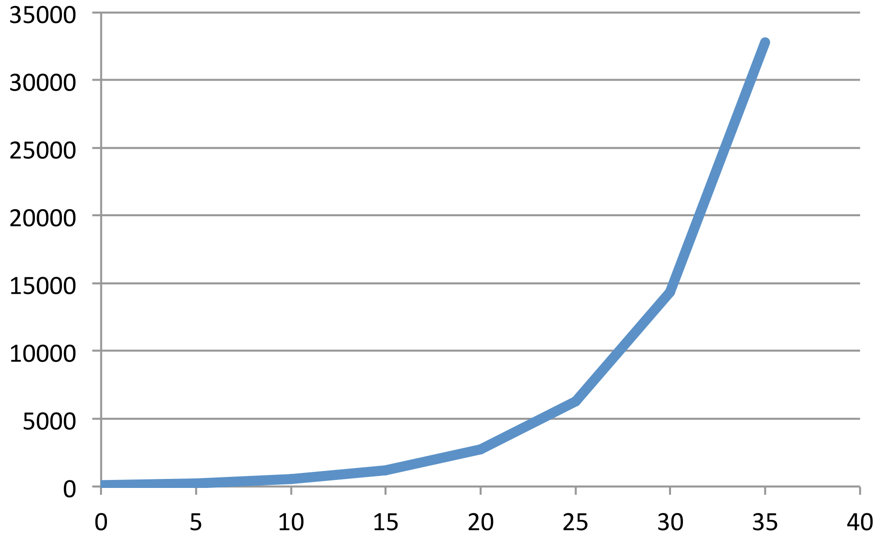
Here is some data…Year on the left, bank credit card dept growing at 18% per year in the middle and $Log_2$ of debt on the right.

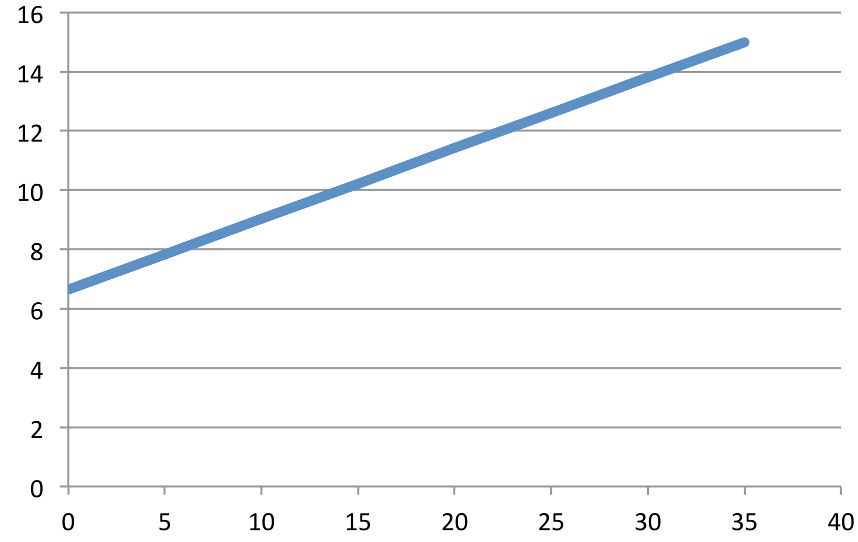
| Year | Money | $Log_2$(Money) |
|---|---|---|
| 0 | 100 | 6.64 |
| 5 | 228 | 7.84 |
| 10 | 523 | 9.03 |
| 15 | 1197 | 10.22 |
| 20 | 2739 | 11.42 |
| 25 | 6266 | 12.61 |
| 30 | 14377 | 13.81 |
| 35 | 32799 | 15.00 |

# Log turns exponential growth into linear growth

On the left is the "raw data" (years vs. debt)

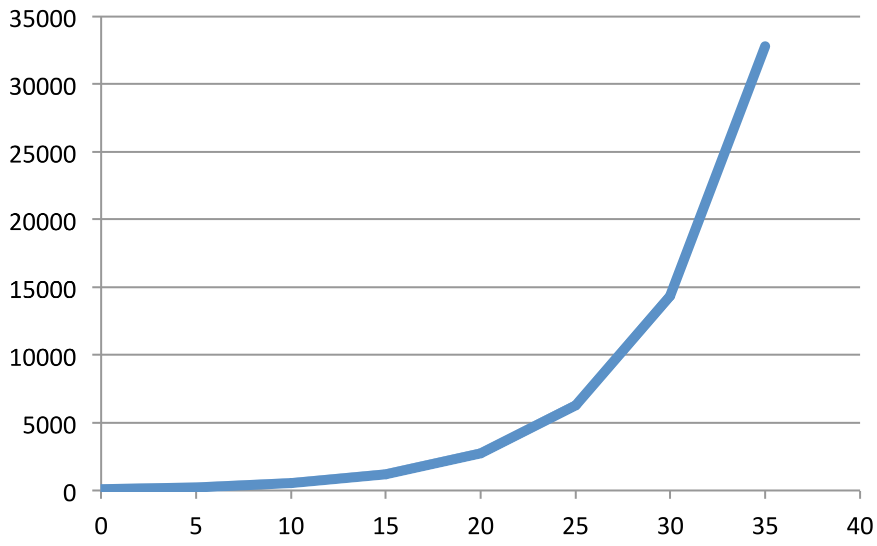
On the right is $\text{Log}_2$ (years vs. $\text{Log}_2$(debt))

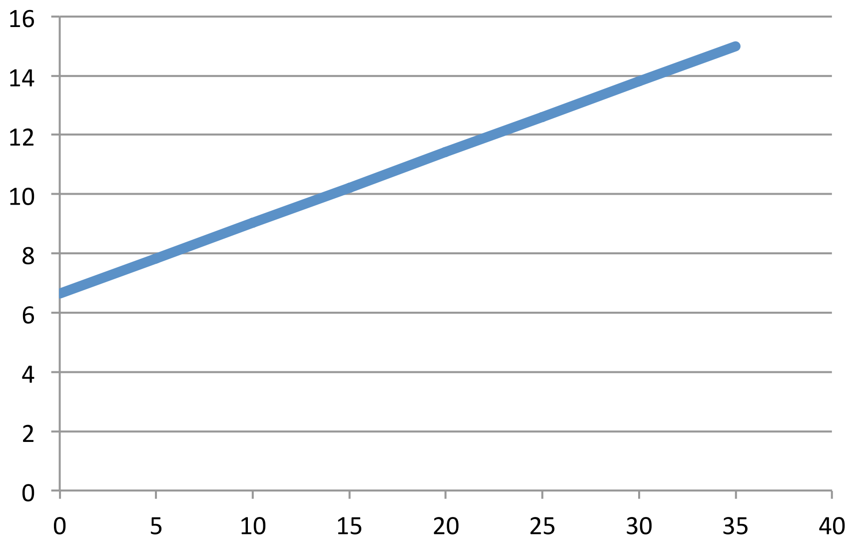


# Log turns exponential growth into linear growth

So we can identify exponential growth by taking Logs…



# Hurricane Bill

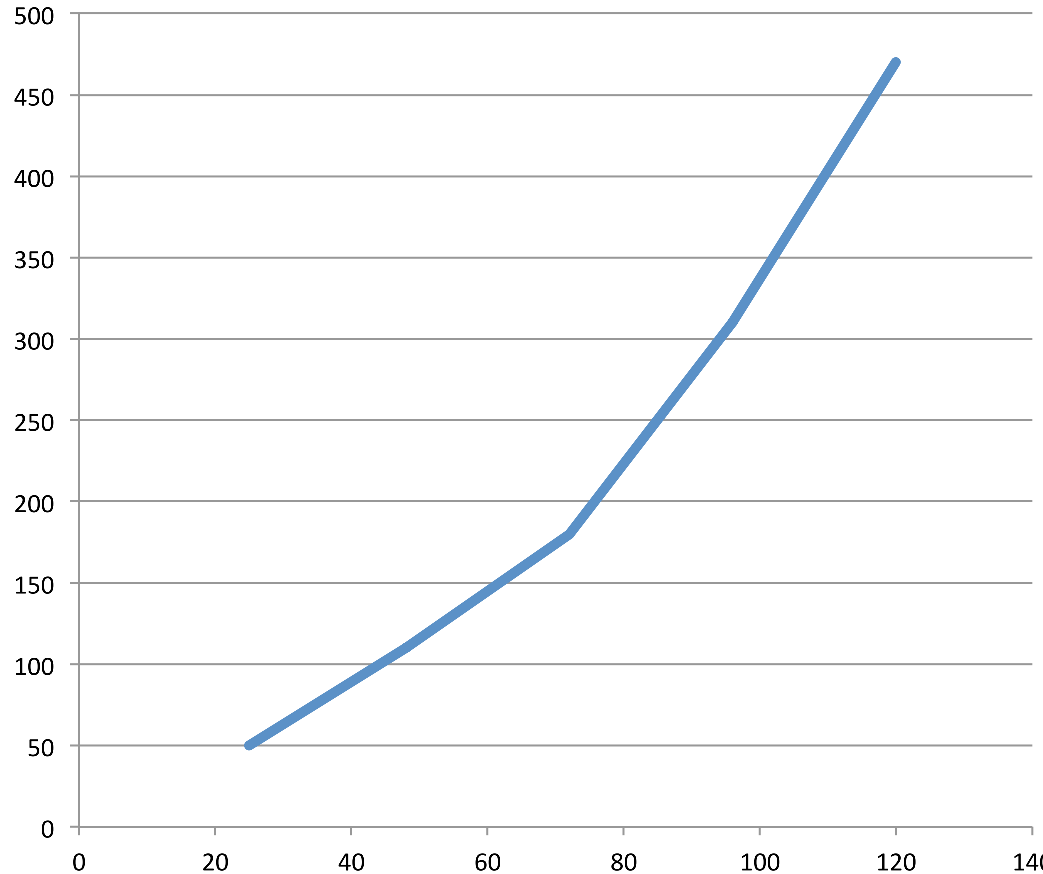
Hurricane Bill came close to New England in August of 2009. Predicting the location of "land fall" of the center of the storm is important since the highest winds occur near the center.

Dr. Jeff Masters of www.wunderground.com wrote a blog post on the accuracy of prediction of the location of the land fall as a function of how far in the future the prediciton was made.

(see: http://www.wunderground.com/blog/JeffMasters/archive.html for August 2009(

# Error in prediction of Bill's landfall

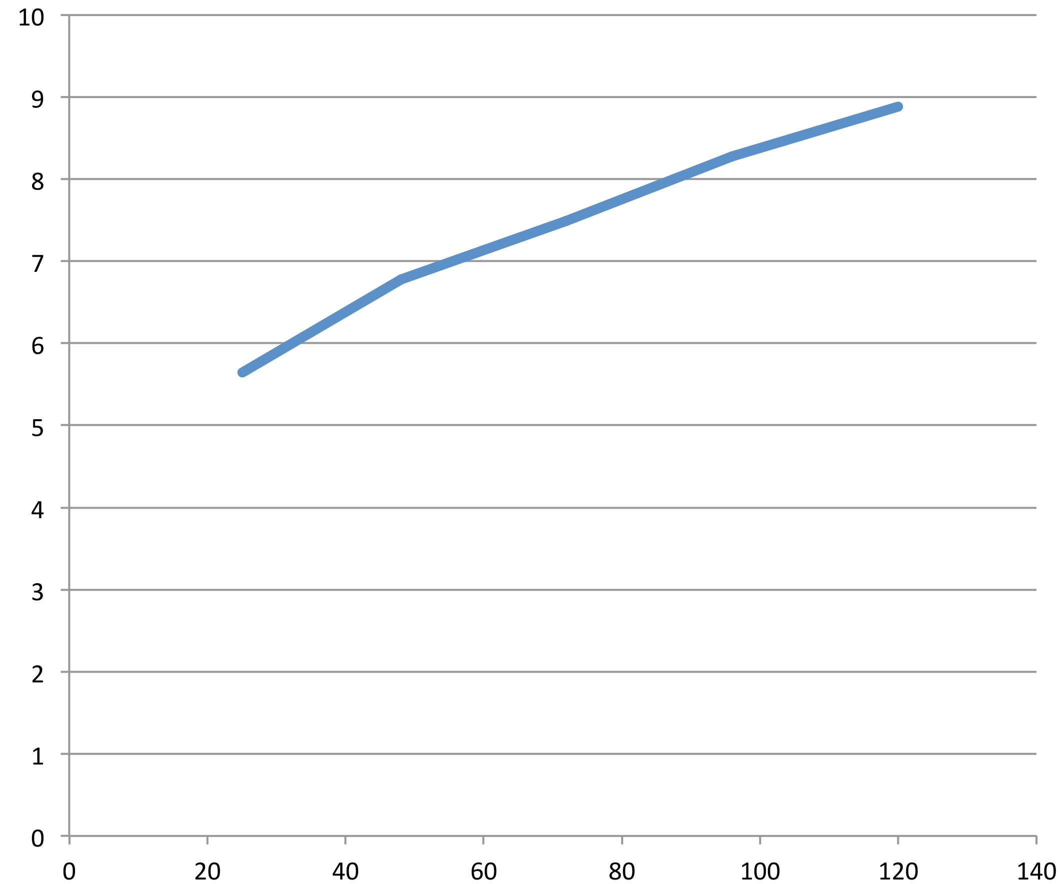
| Hours in future | Nautical miles of error |
|---|---|
| 25 | 50 |
| 48 | 110 |
| 72 | 180 |
| 96 | 310 |
| 120 | 470 |




Is This Exponential growth??

# Error in prediction of Bill's landfall

| Hours | Error | Log base 2 of error |
|---|---|---|
| 25 | 50 | 5.64 |
| 48 | 110 | 6.78 |
| 72 | 180 | 7.49 |
| 96 | 310 | 8.28 |
| `10 | 470 | 8.89 |




~ Line—so yes, exponential…

# Warning!

This is not a lot of data…and remember, any 2 points determine a line. If we only had two data points then of course they would make a line!

Also, we are dealing with “real” data, so we can only expect it to be approximately a line even at best. We’d like a measure of how close the data is to being a line…

(If these questions sound interesting…more math should be in your future.)

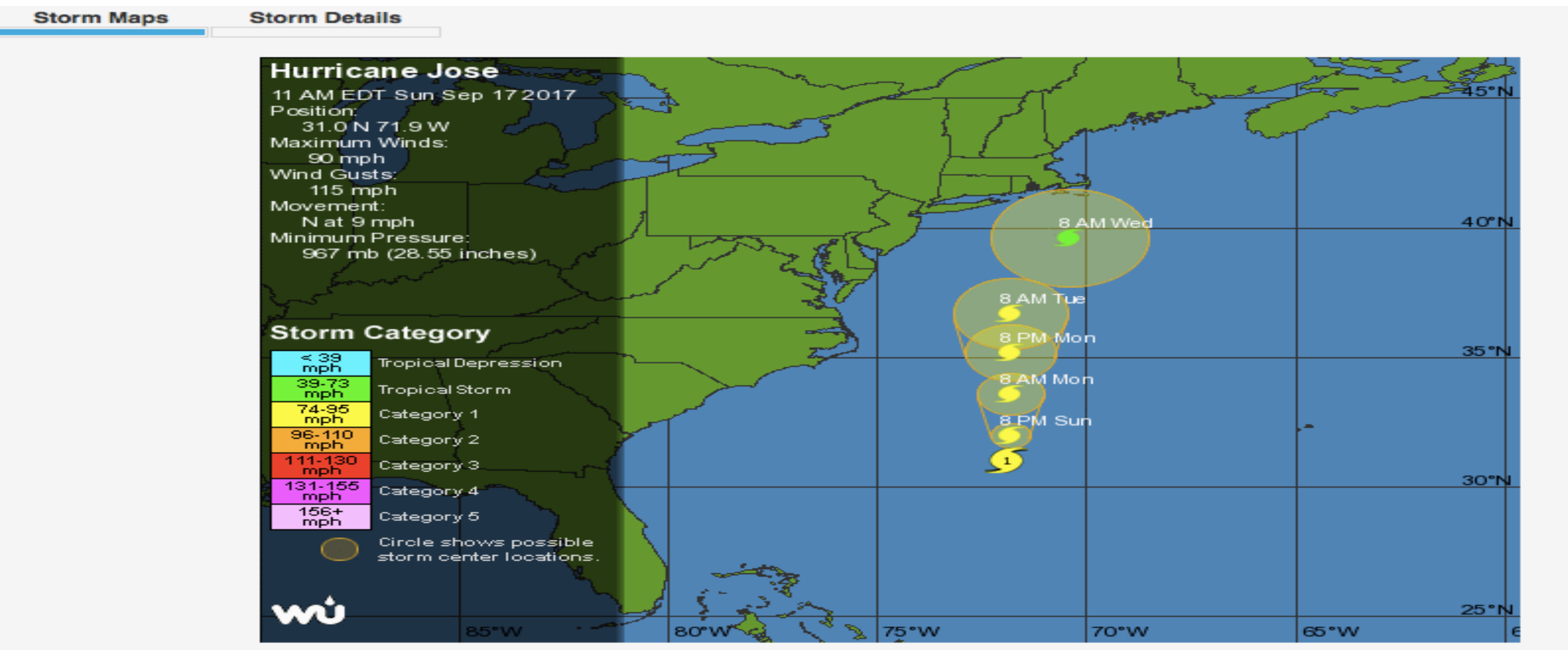
# Hurricane prediction

We will see next week, that this exponential growth of error is not that uncommon…we complain about weather prediction, but we should be amazed at how good it is over the short term, not the errors over the long-term.

We should also be alert…small errors over the short term can grow exponentially.

# Watch for "Cone of uncertainty"

The center dot is just one of the likely positions…




From https://www.wunderground.com/hurricane/atlantic/2017/hurricane-jose

# One more example

Note:  I am not a financial advisor, nor should anyone take any sort of financial advice from me about any investment!!!

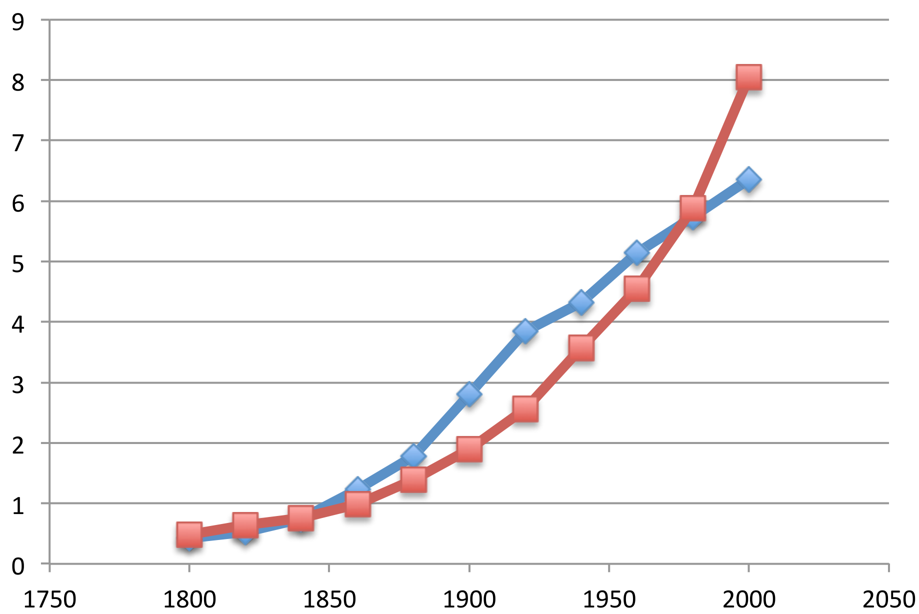
# Where should you buy land…

Land gets more expensive when it is in demand…a small plot of land (no house) in the Boston suburbs can cost $750,000 or more…that can buy 75 acres of high quality North Carolina farm land.

So where is a good place to buy land? Not where the land is already scarce, but where it is going to become scarce.

# Growth of Massachusetts and North Carolina populations

The graphs below are th populations of Massachusetts and North Carolina from 1800 to 2000 (U.S. Census figures). (Blue Mass, Orange NC.)

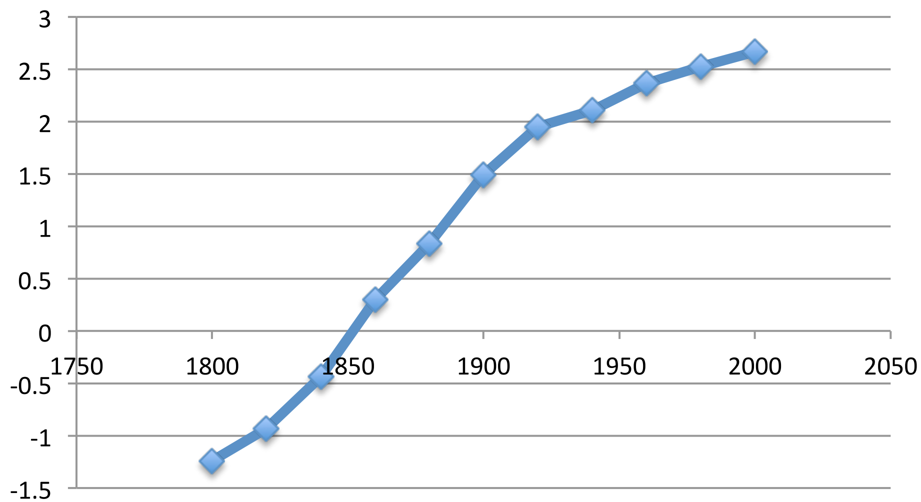


Both populations grow a great deal from about 450,000 in 1800 to 6.3 million for Massachusetts and 8 million for North Carolina…is this exponential growth??

To find out, we take $\mathrm{Log}_2$ of the populations and plot these number

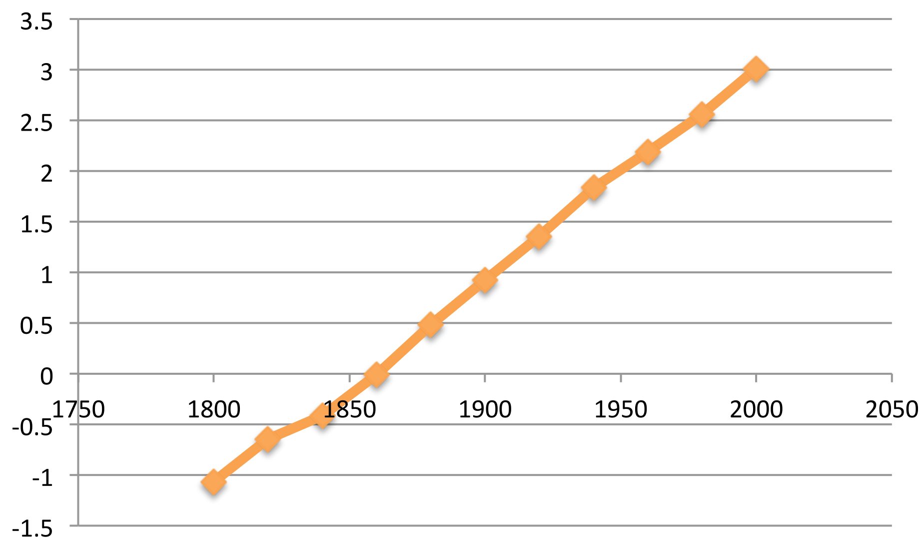
For example population of MA and NC in 1800 was about 450,000 or 0.45 million people. $\mathrm{Log}_2(0.45)=-1.15$, … in 2000 MA population was about 6,000,000 or 6 million people and NC was about 8 million people and $\mathrm{Log}_2(6)=2.58$, and $\mathrm{Log}_2(8)=3$

Recall Logs turn exponential growth into linear growth…so if the populations are growing exponentially, we expect the Log of the population to give a line…for Massachusetts…




Not very much like a line…turns down. Growth seems to be slowing down…

# But North Carolina




The graph of $\text{Log}_2$ of the population is a line! So the population of North Carolina is growing exponentially! (If I were a financial advisor, I'd say…Buy land in North Carolina…)

# By the way…

We haven't given any formula for exponential growth yet…We will do that in the context of a population growth model…

# A Mathematical Model
# in
# Population Ecology…

# Population Models

We want to do one more problem in “Applied” math…There are many, many examples we could choose, but we want to choose one in which we can “start from scratch”.

Choose “mathematical ecology” and look at simple population models.

Turns out this will also introduce us to some very new mathematics!

# Template for Doing Mathematics

Problem

|

Model--------------Repeat---------------------|

| Modify

Examples/Conjectures Model

| |

Proof---Did we answer the question?---No

Yes—Fame $$$

# First Example Problem

As our first example, we take a the population of rabbits in Australia.  Rabbits were introduced for food and sport in the late 1700's but only spread rapidly after a release in 1859.

There were few predators, few diseases and food was available, so the rabbits started to reproduce…well, like rabbits!

Problem:

Predict the future population of an (invasive or new) species (like rabbits in Australia) with few predators and plenty of food.

Model: Biology(!)

Assumptions (This is the "model building" step of our template. We use (very) simple biology).

Assumptions:

1. The number of rabbits at the start of next year is the number of rabbits at the start of this year minus the number that die plus the number that are born.
2. The number of rabbits that die in a given year is proportional to the number of rabbits alive at the start of the year.
3. The number of rabbits born in a given year is proportional to the number of rabbits alive at the start of the year.

To make this useful we have to write it “precisely” in a way that we can make computations…

First, some notation…our assumptions refer to the “number of rabbits alive at the start of a year”— this is a long phrase. Let’s use “rabbits this year” for “number of rabbits alive at the start of this year” and “rabbits next year” for “number of rabbits alive at the start of next year”

We will also use the equals sign “=“ for the verb “is” as well as “+” for “plus” and “-” for minus.

Assumption 1 says:

Rabbits next year = rabbits this year – rabbits that die this year + rabbits that are born this year.

Assumption 2 says;

Rabbits that die this year “is proportional to” the number that are alive at the start of the year (“rabbits this year”)

“Proportional to “ means “a constant times”

So Assumption 2 says

Rabbits that die this year =

constant x (rabbits this year)

Let’s call this constant d for “death rate” constant.

So

Rabbits that die this year = d x rabbits this year

Assumption 3 says sort of the same thing:

Rabbits born this year is proportional to rabbits this year or

Rabbits born this year = b x rabbits this year

where b is the "birth rate" constant.

Putting this together, we get

Rabbits next year

=Rabbits this year – d x rabbits this year

+ b x rabbits this year

That is

Rabbits next year = (1 – d + b) Rabbits this year

If d is bigger than b, then 1-d+b<1 and the rabbits next year is only a fraction of the rabbits this year.

But, if b is larger than d (birth rate bigger than death rate) then

$(1 – d + b) >1$

And the rabbits next year is larger by the factor $(1 – d + b)$ than the rabbits this year. Let's write this number more efficiently.

Let $k=(1 – d + b)$ and call this number the "Growth rate constant".

# Model

Our model is

Rabbit next year = k x Rabbits this year.

This is still kind of clunky…Let's get better notation. Let

R(N)= number of rabbits at the start of year N

So R(0)= number of rabbits at the start of year 0

(say the year rabbits were first released).

So

R(1)=number of rabbits at the start of year 1

R(2)=number of rabbits at the start of year 2

And so on…

If this is year t then "rabbits this year" = R(N) and "rabbits next year" = R(N+1)

# Model

Our model, with all this notation, is

$$R(N+1) = k\,R(N)$$

(in words, rabbits next year is k times rabbits this year).

We are assuming the population grows, so we are assuming $k>1$.

# What does the model predict?

The model says that if we know R(0) (rabbits at year zero) then we can figure out R(1)

$$R(1) = kR(0)$$

(provided we know k.  R(0) is determined by how many rabbits are released, k is determined by rabbit biology and the environment.)

Knowing R(1), we can predict R(2)

$$R(2) = kR(1)$$

But wait,

$R(1) = kR(0)$ and $R(2) = kR(1)$

Gives

$R(2)= k\ (k(R(0)) = k^2\ R(0)$

Similarly

$R(3) = kR(2)$

So

$R(3) = kR(2) = k\ (k^2\ R(0))= k^3\ R(0)$

In general, if

$$R(N) = k^N R(0)$$

Then

$$R(N+1) = kR(N) = k\,(k^N R(0)) = k^{(N+1)} R(0).$$

So, we have established that

$$R(N) = k^N R(0)$$

For all N = 1,2,3,… forever.

(This is an example of a "proof by induction"—

Show the pattern holds at the start and show it keeps holding, then it must always hold.)

# Exponential Growth!

This is the general form of the formula for exponential growth—the variable (time in this case in years) is in the exponent(!).

$R(N) = k^N R(0)$

Notice that we need a starting value (R(0)) —"exponential" means rate of growth…need to know where to start.