MA/CS 109 Lecture 6
Exponential functions

Exponential functions have “explosive growth”
Example: \( f(t) = 2^t \)

Growth in one time unit
\[
f(t+1)-f(t) = 2^{(t+1)}-2^t = 2^t (2-1)=2^t=f(t)
\]

So the rate of growth is proportional to the value of the function! Exponential growth has the property that the bigger the function, the faster it grows!
Consider the function $f(t) = 2^t$

Powers of 2:

- $f(1) = 2$
- $f(2) = 2^2 = 4$
- $f(3) = 2^3 = 8$
- $f(4) = 2^4 = 16$
- $f(5) = 2^5 = 32$
- $f(6) = 2^6 = 64$
- $f(7) = 2^7 = 128$
- $f(8) = 2^8 = 256$
- $f(9) = 2^9 = 512$
- $f(10) = 2^{10} = 1024$

Memorize!...will be very useful for computer science. In particular, $2^{10}$ is about 1000.
Exponential growth is faster than quadratic...

Blue linear(3t), orange quadratic (t^2), green exponential (2^t)
Exponential growth is a lot faster!!

Blue linear(3t), orange quadratic (t^2), green exponential (2^t)
Most important thing you Ever Learn:
Exponential growth can not go on forever.
D’uh:

In a finite environment, unbounded growth cannot go on forever no matter what the rate of growth...
What is different about exponential growth is that you reach the limit of growth “suddenly”...

Examples:

Limit 100. Linear growth 10t.
Reaches limit at t=10
Quadtratic growth:

Limit 100. Quadratic growth $t^2$.
Reaches limit at $t=10$
Exponential Growth

Limit 1000. Exponential growth $2^t$.
Reaches limit at $t=10$
All together:

Note how exponential growth both gets larger and growth rate accelerates as $t$ gets larger!
So most of the growth takes place quickly at the end.
This is even more dramatic for larger times...

Same functions up to t=20, limit 1,000,000

Note these are not much different for t below 10 or 12 (on this scale...)
This growth of the growth rate is a property of exponential growth, no matter what type—

Use of fossil fuels grows around 2% per year. So the more fossil fuels we use this year, the more growth for next year...but 2% isn’t much...right?

This means that the amount of fossil fuels used this year is

\[
\text{Amount last year} + \text{Amount last year} \times 0.02 = \text{Amount last year} \times 1.02
\]
So if $F(0)$ is used in year 0

Amount used year 1 = $F(1) = 1.02 \times F(0)$

Amount used year 2 = $F(2) = 1.02 \times F(1)$

\[ = 1.02 \times (1.02 \times F(0)) \]

\[ = 1.02^2 \times F(0) \]

Amount used year 3 = $F(3) = 1.02 \times F(2)$

\[ = 1.02 \times (1.02^2 \times F(0)) \]

\[ = 1.02^3 \times F(0) \]

And so on...
Still exponential growth...

Still the same shape
(In fact, doubles every 35 years).
So exponential growth is dangerous because there isn’t much time between abundance and shortage...the end comes quickly.

There are many examples of this in economics—"economic bubble".

Tulip mania—in 1634-37 the price of tulip bulbs doubled 8 times. But exponential growth can not go on forever...by 1638 the price had fallen to below 1634 prices—a “crash”.
Housing prices 2000-2006...then crash...

There are many other examples of markets that grew exponentially—then stopped growing exponentially with a “correction” or a “crash”.

Note...Exponential growth doesn’t necessarily lead to a crash—the number of email servers grew exponentially for ... but then the rate of growth slowed. It is still growing, just not as fast as before.
Even when there is a crash—the world doesn’t usually end...

But exponential growth can’t go on forever implies that something must change. When radical change happens, it is always good for some and bad for others...sometimes very good for a few and very bad for many.

Moral: When you see something growing exponentially, don’t expect it to last forever and expect change.
Good Exponential Growth

Croissant are made by creating a layer of butter surrounded by dough. Then stretching and folding the dough until there are many layers...

Each fold triples the number of layers of butter, start with 1 layer of butter, then 3 layers, then 9 layers, then 27 layers then 81 layers...few bakers go beyond this...why?
One more type of function...

There are functions that grow even faster than exponentially... for example

if $f(t) = 2^t$ then $s(t) = 2^{f(t)}$ or $2$ to the $2$ to the $t$ grows even faster!

There are functions that grow more slowly even than linear functions...
Recall progressive family C

In this family each generation is twice the size of the previous generation and at holiday season each generation gets together and gives one substantial gift to a charity in the name of the family.

The oldest generation had 2 members—1 gift. The next generation had 4 members (so a total of 6 members of the family)—2 gifts
The next generation has 8 members for a total of 14 members—and 3 gifts
The next generation has 16 members for a total of 30 members—and 4 gifts

The number of gifts is growing very slowly—only as fast as the number of generations (which is much less than the number of people in the family).
This is definitely a different sort of growth. Here the rate of growth decreases as the numbers get bigger.
Log functions

The type of function described above is called a Logarithm function... We will only use one type of Log function—Log base 2, denoted Log$_2$(N).

Definition: Log$_2$(N) is the number you raise 2 to in order to obtain N.

So Log$_2$(4) = 2 because $2^2=4$

Log$_2$(8) = 3 because $2^3=8$

And so on...
So $\log_2(64) = 6$ because $2^6 = 64$

And

$\log_2(1000)$ is approximately 10 because $2^{10}$ is about 1000.

$\log_2(2000)$ is about 11 because $2^{11}$ is about 2000.
Property of Logs

The property of $\log_2$ we will need is that $\log_2$ grows so slowly, that it can turn exponential growth into linear growth...

This is a property of Log’s that you may remember from study for the SAT’s

\[ \log_2(a^t) = t \log_2(a) \]

That is Log’s turn exponentiation in multiplication.
Suppose we have data...

<table>
<thead>
<tr>
<th>t</th>
<th>F(t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
</tr>
<tr>
<td>10</td>
<td>202</td>
</tr>
<tr>
<td>15</td>
<td>2862</td>
</tr>
<tr>
<td>20</td>
<td>40642</td>
</tr>
<tr>
<td>25</td>
<td>577062</td>
</tr>
<tr>
<td>30</td>
<td>8193465</td>
</tr>
</tbody>
</table>
Graph looks like exponential growth...
But if we take the $\log_2$ of the data

<table>
<thead>
<tr>
<th>$t$</th>
<th>$F(t)$</th>
<th>$\log_2(F(t))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.7</td>
<td>0.76</td>
</tr>
<tr>
<td>5</td>
<td>14</td>
<td>3.83</td>
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<tr>
<td>10</td>
<td>202</td>
<td>7.65</td>
</tr>
<tr>
<td>15</td>
<td>2862</td>
<td>11.48</td>
</tr>
<tr>
<td>20</td>
<td>40642</td>
<td>15.31</td>
</tr>
<tr>
<td>25</td>
<td>577062</td>
<td>19.14</td>
</tr>
<tr>
<td>30</td>
<td>8193465</td>
<td>22.97</td>
</tr>
</tbody>
</table>
Graphing the $\log_2$ of the data the exponential growth becomes linear growth...

So $\log_2$ turns exponential growth into linear growth.
This is very useful for recognizing exponential growth...

Two examples:

Error in the prediction of land fall of a hurricane

<table>
<thead>
<tr>
<th>Hours in the future</th>
<th>Error (miles)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>50</td>
</tr>
<tr>
<td>48</td>
<td>110</td>
</tr>
<tr>
<td>72</td>
<td>180</td>
</tr>
<tr>
<td>96</td>
<td>310</td>
</tr>
<tr>
<td>120</td>
<td>470</td>
</tr>
</tbody>
</table>
Growth...is it exponential growth...

Use the log of data
<table>
<thead>
<tr>
<th>Hours in the future</th>
<th>Error (miles)</th>
<th>( \log_2(\text{Error}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>50</td>
<td>5.64</td>
</tr>
<tr>
<td>48</td>
<td>110</td>
<td>6.78</td>
</tr>
<tr>
<td>72</td>
<td>180</td>
<td>7.49</td>
</tr>
<tr>
<td>96</td>
<td>310</td>
<td>8.28</td>
</tr>
<tr>
<td>120</td>
<td>470</td>
<td>8.88</td>
</tr>
</tbody>
</table>
Graph of the $\log_2$ of the data...pretty close to a straight line—so the original data is exponential growth.

So error in hurricane prediction grows exponentially with distance into the future of the prediction.