MA/CS 109 Lecture 7

Back
To
Exponential Growth Population Models
Homework this week

1. Due next Thursday (not Tuesday)
2. Do most of computations in discussion next week
3. If possible, bring your laptop (or Excel capable machine) to discussion.
4. Check the course web page before discussion for “pre-lab reading”..
Template for Doing Mathematics

Problem

| Model----------------Repeat-----------------------------|
| Modify |

Examples/Conjectures | Model |

Proof---Did we answer the question?---No

Yes—Fame $$$
“Rabbits in Australia”

Build a model to predict the future population of a species that currently has a small number of individuals. The environment is assumed to be large, friendly, and have lots of food...

Expect fast growth—How fast?

First the biology...
Assumptions:

1. The number of rabbits at the start of next year is the number of rabbits at the start of this year minus the number that die plus the number that are born.

2. The number of rabbits that die in a given year is proportional to the number of rabbits alive at the start of the year.

3. The number of rabbits born in a given year is proportional to the number of rabbits alive at the start of the year.
Last time, turned this into an Equation

Let \( R(N) = \text{Rabbits at time N} \)

(Units: We’ll use time in years
We can measure the number of rabbits by recording the number of rabbits, or how many thousands or millions of rabbits, or even tons of rabbits...so if \( R(N) \) is a fraction or decimal, it doesn’t mean we have pieces of rabbits...)
Using the assumptions, we got to...

Rabbits next year

\[ = \text{Rabbits this year} - d \times \text{rabbits this year} \]
\[ + b \times \text{rabbits this year} \]

(d = death rate, b = birth rate)

That is

Rabbits next year = (1 – d + b) Rabbits this year

If d is bigger than b, then 1-d+b<1 and the rabbits next year is only a fraction of the rabbits this year.
But, if $b$ is larger than $d$ (birth rate bigger than death rate) then

$$(1 - d + b) > 1$$

And the rabbits next year is larger by the factor $(1 - d + b)$ than the rabbits this year. Let’s write this number more efficiently.

Let $k = (1 - d + b)$ and call this number the “Growth rate constant”.

Model

Our model, with all this notation, is

\[ R(N+1) = k \cdot R(N) \]

(in words, rabbits next year is \( k \) times rabbits this year).

We are assuming the population grows, so we are assuming \( k > 1 \).
What does the model predict?

The model says that if we know $R(0)$ (rabbits at year zero) then we can figure out $R(1)$

$$R(1) = kR(0)$$

(provided we know $k$. $R(0)$ is determined by how many rabbits are released, $k$ is determined by rabbit biology and the environment.)

Knowing $R(1)$, we can predict $R(2)$

$$R(2) = kR(1)$$
But wait,

\[ R(1) = kR(0) \text{ and } R(2) = kR(1) \]

Gives

\[ R(2) = k (kR(0)) = k^2 R(0) \]

Similarly

\[ R(3) = kR(2) \]

So

\[ R(3) = kR(2) = k (k^2 R(0)) = k^3 R(0) \]
In general, if
\[ R(N) = k^N R(0) \]
then
\[ R(N+1) = kR(N) = k (k^N R(0)) = k^{(N+1)} R(0). \]

So, we have established that
\[ R(N) = k^N R(0) \]
for all \( N = 1, 2, 3, \ldots \) forever.
(This is an example of a “proof by induction”—Show the pattern holds at the start and show it keeps holding, then it must always hold.)
Exponential Growth!

This is the general form of the formula for exponential growth—the variable (time in this case in years) is in the exponent(!).

\[ R(N) = k^N R(0) \]

Notice that we need a starting value \( R(0) \)—”exponential” means rate of growth...need to know where to start.
Got the Model:
Now Predict the Future!

In order to use our model

\[ R(N+1) = kR(N) \]

And its predictions for population

\[ R(N) = k^N R(0) \]

We need to know two things:

The value of the growth rate constant \( k \)

(That is, information about rabbits and environment)

The value of \( R(0) \)

(The number of rabbits we start with.)

Well, nothing is for free... But with this little bit of information we get the ENTIRE future... not bad!!
Growth rate constant:

Start with some examples...
What is a reasonable value for $k$?

Well, google says 1 to 14 babies per litter with gestation period of a month...so the birth rate could be quite high!
But life is hard for rabbits.. life span less than 3 years and many don’t make it to maturity.

Let’s start with $k=2$
What about $R(0)$. Technically we can’t start with 1 rabbit and just 2 seems dangerous...let’s start with $R(0)= 30$ rabbits

(and $R(N)$ will be actual numbers of rabbits alive at the start of year N)

So $R(0)=30$

$R(1)= 2 \times R(0)=2\times30=60$

$R(2)= 2 \times R(1)=2\times(2\times R(0))=4\times R(0)=4\times30=120$
The Future...

In general $R(N) = 2^N \times 30$

So $R(10) = 2^{10} \times 30 \sim 1000 \times 30 = 30,000$
Lots of rabbits, but

$R(11) = 2 \times R(10) \sim 60,000$
(lots more)

And $R(20) = 2^{20} \times 30 = 2^{10+10} \times 30 = 2^{10} \times 2^{10} \times 30$

$\sim 1000 \times 1000 \times 30 = 30,000,000$
After 20 and 50 years

Note the shape—exponential explosion.
These are unbelievably huge numbers...after 10 years, “only” 30 thousand rabbits, after 20 years, 30 million rabbits but after 50 years, 8000000000000000 rabbits...

We learn 3 things from this model...first, the rabbit population grows slowly at first, then the rabbit population grows much much faster, then the rabbit population gets truly unbelievably big.

(The population prediction after 50 years is more than one rabbit per square foot of Australia.)
And after 100 years? Just silly...

So our model might be pretty accurate for small populations, but is nowhere close when the population gets large.

Next we will alter the model (in 2 different ways) to take into account factors that can limit growth...
But two more things about Exponential Growth

But exponential growth has a surprise...
Suppose we had the same $k=2$ value but instead of having 30 rabbits at time zero, we had 32 rabbits. Tiny error in our initial condition...

Then after 10 years we would have

$$R(10) = 2^{10} \times 32 \approx 32,000$$

The error of 2 rabbits at time zero has caused an error of 2000 rabbits at time 10.
After 20 years, 32 rabbits becomes
\[ R(20) = 2^{20} \times 32 = 2^{10} \times 2^{10} \times 32 \approx 1000 \times 1000 \times 32 \]
\[ = 32,000,000 \text{ (compared to 30,000,000 when starting with 30)} \]
So after 20 years the error of 2 rabbits at time 0 has grown to 2 million rabbits...

Small error in the initial value grow exponentially too!
Error in prediction for 30 and 32 rabbits at start.

Error in predicted population after 50 years
20000000000000000 rabbits...
Moral

For this model, the error doesn’t really matter so much...the model is predicting a gigantic number of rabbits – so large that a million here, a million there, won’t even be noticed.

But this property that an initial error can grow exponentially turns out to be fundamentally important...remember and stay tuned.
One more way to “picture” exponential growth

We are now used to graphs showing exponential growth—the slow start at small times and then the “explosion”.

This type of graph is called a “time series”, because the horizontal axis is time. We see the prediction for one $R(0)$ far into the future.
Another picture of the model

But recall the form of the model

\[ R(N+1) = k \, R(N) \]

Or “rabbits next year = k x rabbits this year”

For our example above, k=2 so

“rabbits next year = 2 x rabbits this year”

This is a very simple type of function! The variable on the left is just 2 times the variable on the right.
Graph of the model

The graph below has “rabbits this year” on the horizontal axis and “rabbits next year” on the vertical axis. The blue line is the model “rabbits next year = 2x rabbits this year”

While the red is “rabbits next year = rabbits this year” included for comparison.

The value of k gives the “slope” of the blue line.
Graph of the model

This picture lets us predict only one year in the future...Find rabbits this year on the horizontal and then go up to the blue and across to the vertical to get rabbits next year. But, we can make that one year prediction for lots of different values of rabbits this year.
Graph of the model

Looking at this picture, we see that the population of rabbits next year is always bigger than the population this year (the blue line is above the red line) except at rabbits=0 (extinct is forever.) Also, the more rabbits this year, the bigger the growth next year (the farther the blue is above the red.)
Finally, we can adjust the value of $k$ and see what happens to the one year predictions... suppose we were modeling whale populations. Hopefully the population of whales is growing, but it is not growing very fast, so $k$ is much smaller...let’s try $k=1.1$, so just bigger than 1.
So for whales, the growth is still larger when the population this year is larger, but only just a little (the blue line is just above the red line).

We still have exponential growth, but it takes longer for the population to get large so for the explosion to happen.
Quiz...

One of the graphs below is the model for population of and the other for greenland sharks and the other for sardines...which is which?

Populations
Next year

Population

Next year

Population this year

Population this year
Hard Quiz...

The blue graph below represents a modification of the exponential growth population model with $k=2$. What change in how the population changes could give this model? (Red is next year = this year for reference).
Better Models

The model predictions over the short-term are reasonable, but over the long-term are not realistic...if they were, the rabbits would have covered Australia long ago.

People wouldn’t stand for the rabbits pushing them out of Australia...they would do something about it, and they did...
Exponential Growth with Harvesting

Harvesting is “removing” a number of rabbits from the population...non-natural causes.

Add an assumption

Assumption 4: A constant number H of rabbits will be “harvested” each year.

So the population is “decreased” by H in addition to the usual deaths...
How do we adjust the model?

Assumption 4: A constant number $H$ of rabbits will be “harvested” each year.

So the population is “decreased” by $H$ in addition to the usual deaths...

New Model--
Rabbits next year = $k \times$ Rabbits this year − $H$
Or $R(N+1) = k \times R(N) - H$
How does this change the prediction?

First the prediction 1 year into the future?
Suppose we still have $k=2$ and we take $H = 45$.

Old Model
How does this change the prediction?

First the prediction 1 year into the future?
Suppose we still have $k=2$ and we take $H = 45$.

Old Model

New Model
New Model: \( R(N+1) = 2xR(N) - 45 \)

To get the model with harvesting \( H=45 \) per year, slide the blue graph “Down” by 45….but you can’t have negative rabbits, so if the blue graph is below zero, make it zero.
Exponential Growth with Harvesting: $R(N+1) = 2xR(N) - 45$

What does this model predict?

When population this year is large, then population next year is larger than this year...

(The same as without harvesting)
Exponential Growth with Harvesting: $R(N+1) = 2xR(N) - 45$

What does this model predict?
When the population this year is very small then the population next year is zero. Harvesting drove the population to extinction.
Exponential Growth with Harvesting: \( R(N+1) = 2xR(N) - 45 \)

When the population is small, but not too small, the population next year is smaller than the population this year (but not zero), so population decreases.
Exponential Growth with Harvesting:

$R(N+1) = 2 \times R(N) - 45$

When the population is just right between small and big, then the population next year equals the population this year. This is a “fixed point” of population (where blue and red cross).

![Graph showing population growth and harvesting]
That’s one year in the future...

The one year in the future prediction of this model depends very much on the size of the population this year.

What about the “long-term” future?
What does this model predict?
Far Future: \[ R(N+1) = 2 \times R(N) - 45 \]

Very small initial population:

Next year’s population is zero...extinct is forever.
Far Future: \( R(N+1) = 2xR(N) - 45 \)

Small initial population:

Next year’s population is smaller than this year’s and the year after that is smaller still... until the population is very small and then goes extinct.
Far Future: $R(N+1)=2xR(N) - 45$

Large initial population:

Next year’s population is larger than this year’s and the year after that is larger still... and the rate of growth gets bigger with population—exponential growth (even with harvesting!).

![Graph showing exponential growth](image-url)
Far Future: \( R(N+1)=2xR(N) - 45 \)

Just right initial population: (Here Pop.=45)

Next year’s population is equals this years population, so the year after that it stays equal... forever. Population stays fixed (hence the name “fixed point”).
Why is just right 45?

We are looking for the population $P$ where
Next year $= 2 \times$ this year $-45 = $ this year
Or
$2 \times P - 45 = P$
Or
$2xP = P + 45$
Or
$2xP-P = 45$
Or
$P=45$ (where the red and blue cross).
What can we learn from this model?

For $k=2$ and $H=45$ if the population is less than 45 then the creature goes extinct... If the population is greater than 45, the population will blow up exponentially and if it equals 45, population will stay at 45 forever...

What if we change $k$ and/or $H$???