MA/CS 109 Lecture 8

Exponential Growth with Harvesting

And

Limited Growth Models

Trouble with Predictions of the Exponential Growth Model

While the Exponential Growth model makes reasonable (and interesting) predictions for the growth of a small population in a large environment, when the population becomes large the predictions get so big that they become silly…speaking of silly—here is an example of exponential growth…

https://www.youtube.com/watch?v=WXQ0CMp6BI8

Exponential Growth with Harvesting

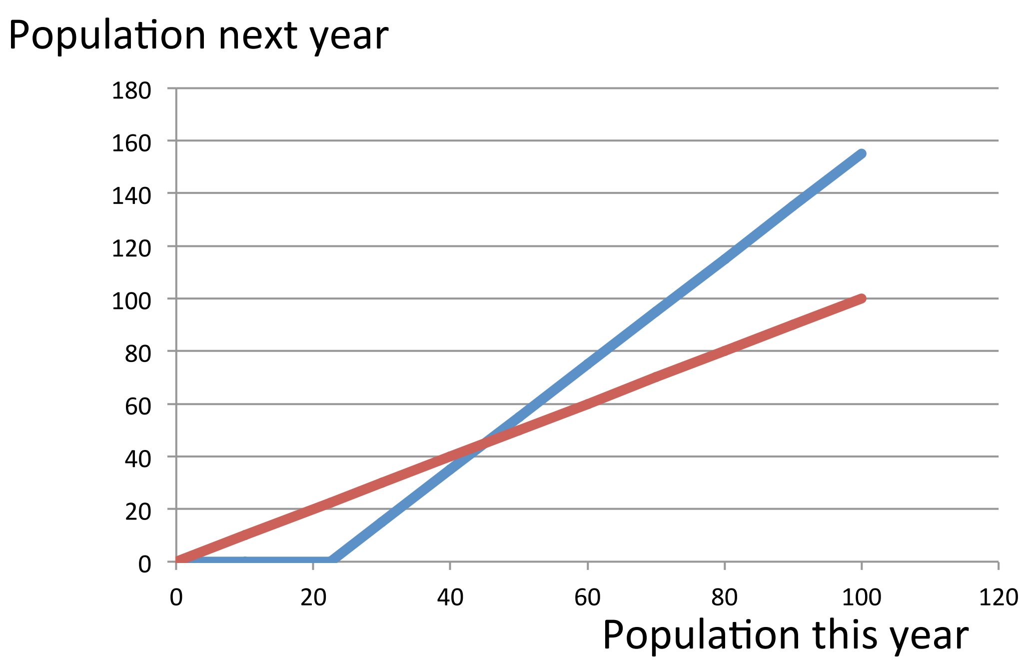
One attempt to modify the model is to add "Harvesting", i.e., we attempt to control the growth of the population by harvesting or removing as many as we can each year. We let H be the "harvesting rate", the number we can remove each year, and we get the new model

$$R(N+1) = k\,R(N) - H$$

(or rabbits next year = k times rabbits this year minus Harvested(H)

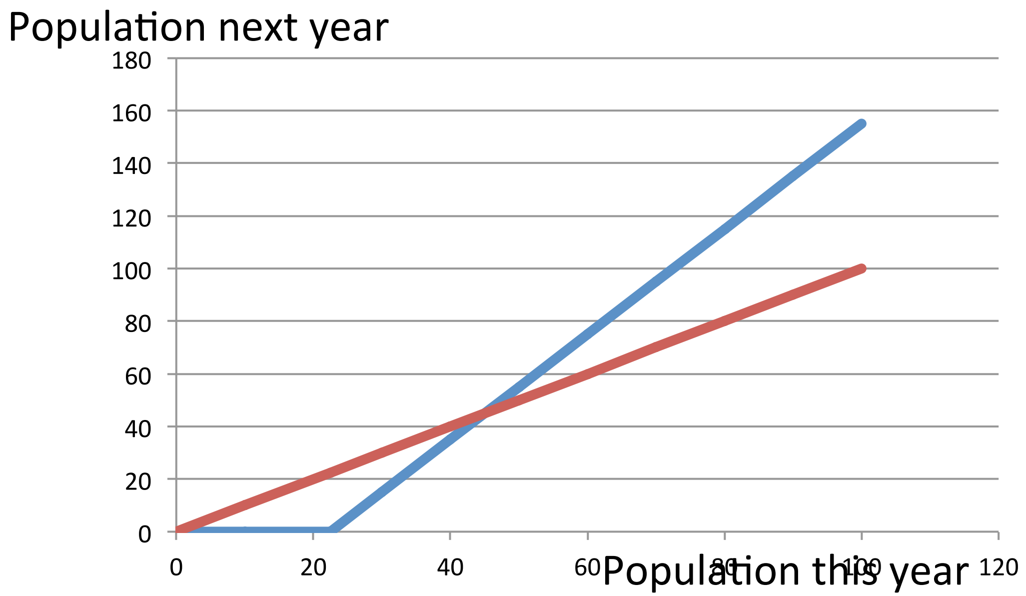
Exponential Growth with Harvesting

The blue graph below is the new model (for k=2, H=45 as an example). From the picture, we see that very small populations go extinct, small populations decrease and large populations grow (and grow faster as they get bigger).



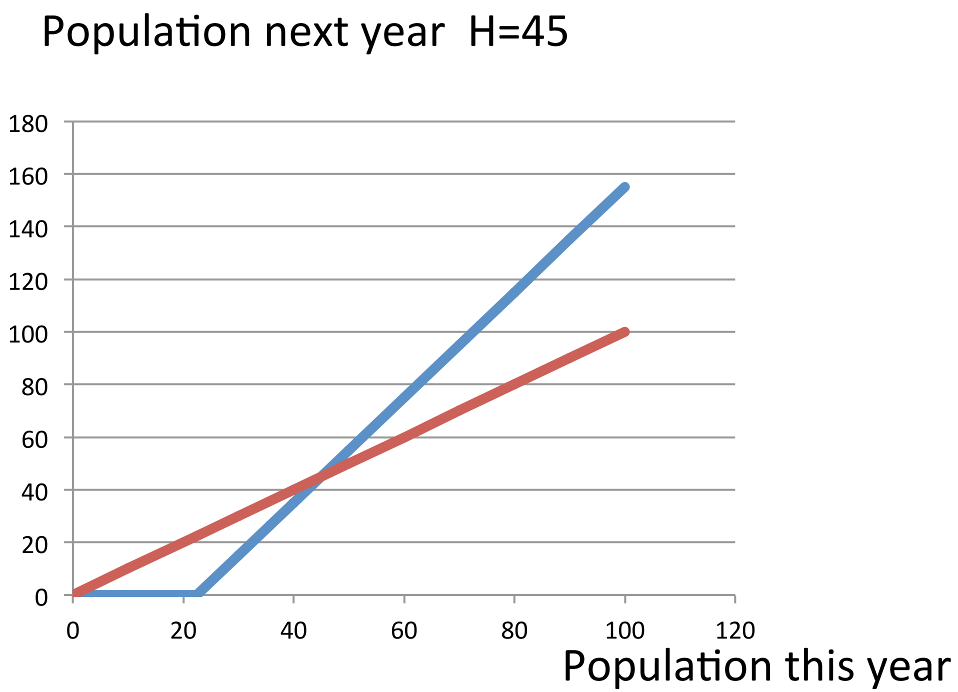
Exponential Growth with Harvesting

Between small and large there is “just right”—a fixed point of the population where the population stays the same from each year to the next.



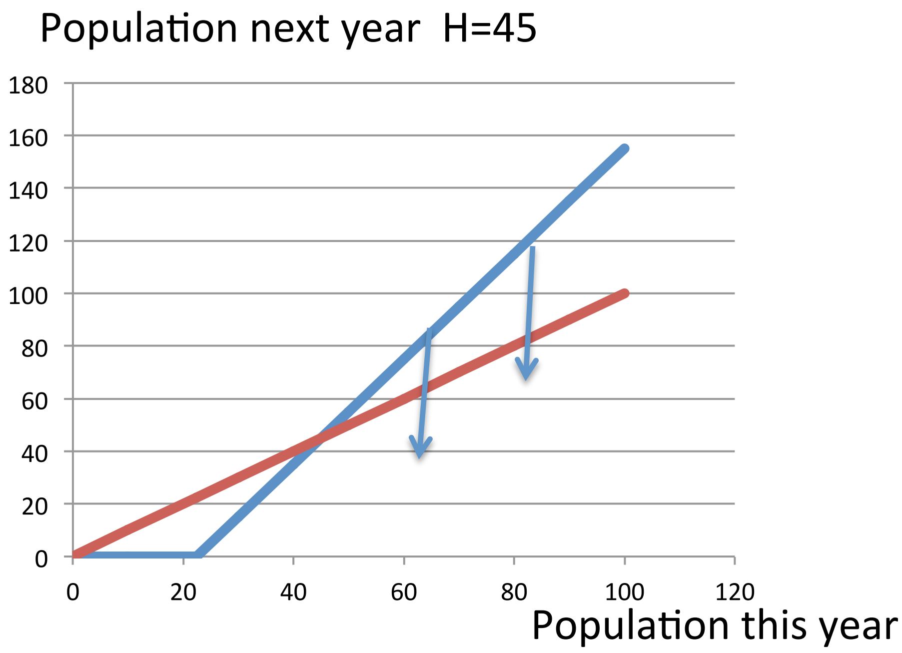
Exponential Growth with Harvesting

The location of the fixed point depends on H.



Exponential Growth with Harvesting

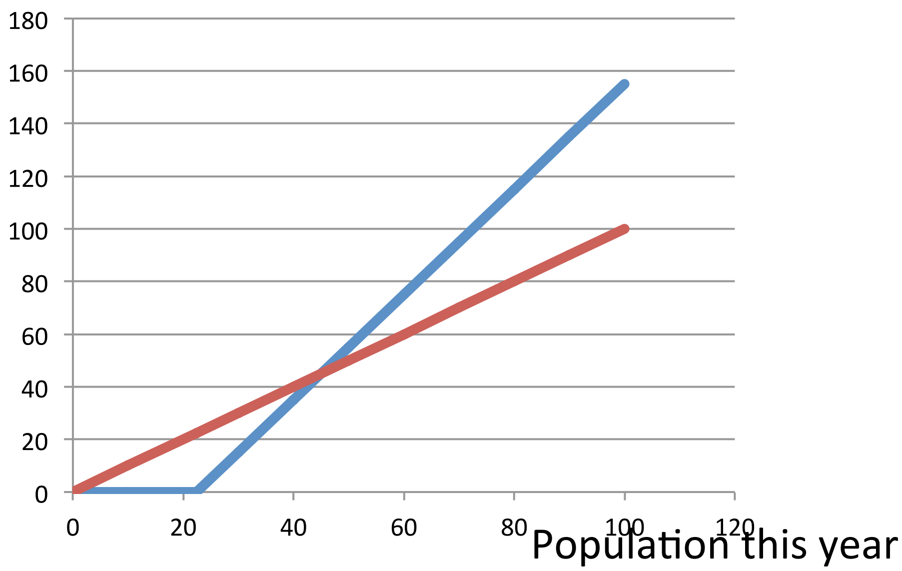
Increasing H corresponds to pushing the graph of the model (the blue graph) down…(but if the blue graph goes below zero we set it to zero since there are no negative rabbits).



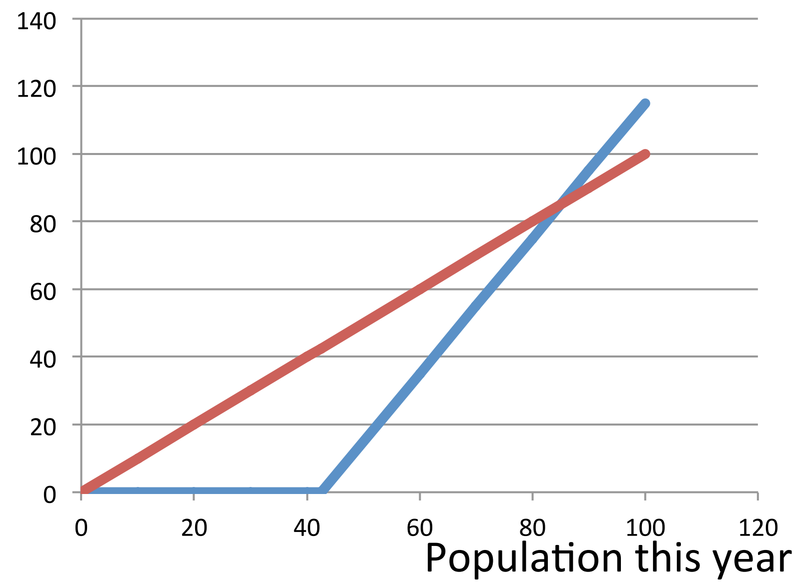
Exponential Growth with Harvesting

The location of the fixed point depends on H. The bigger we make H (keeping k constant), the farther to the right the fixed point is pushed, so the larger the “small” populations that decrease become.

Population next year H=45

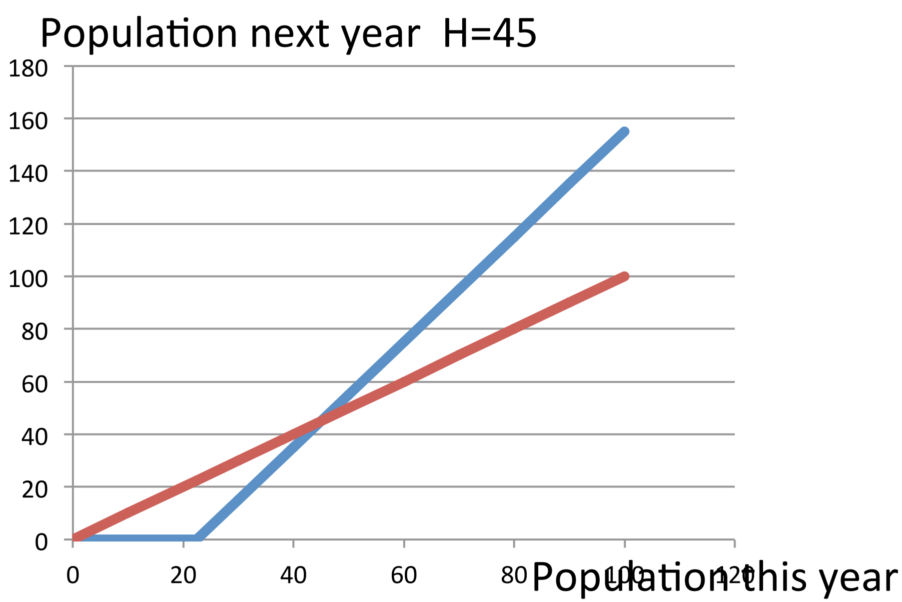


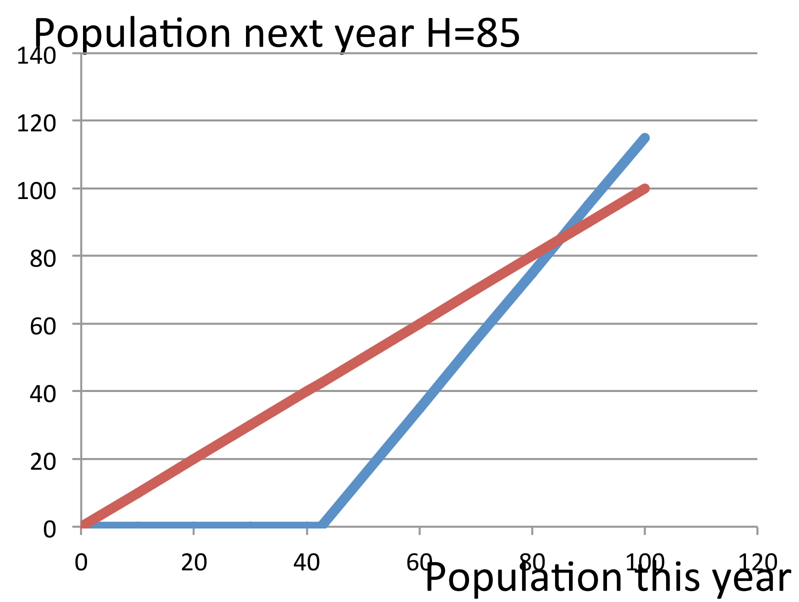
Population next year H=85



Exponential Growth with Harvesting

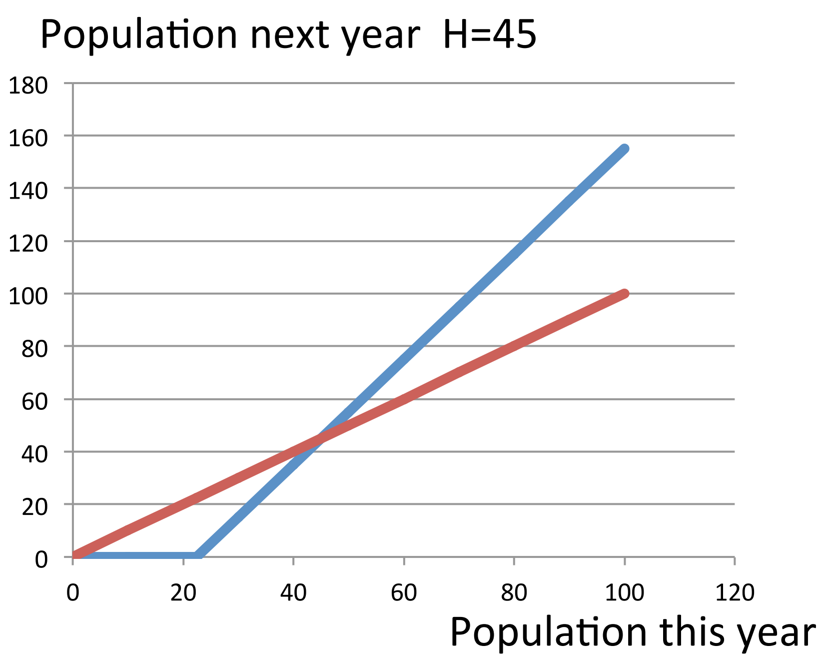
In both these graphs, k=2, but H is larger on the right. The location of the fixed point is 85 when H is 85. Any initial population will decrease if it is less than 85 and eventually go extinct, but larger than 85 will grow (exponentially).

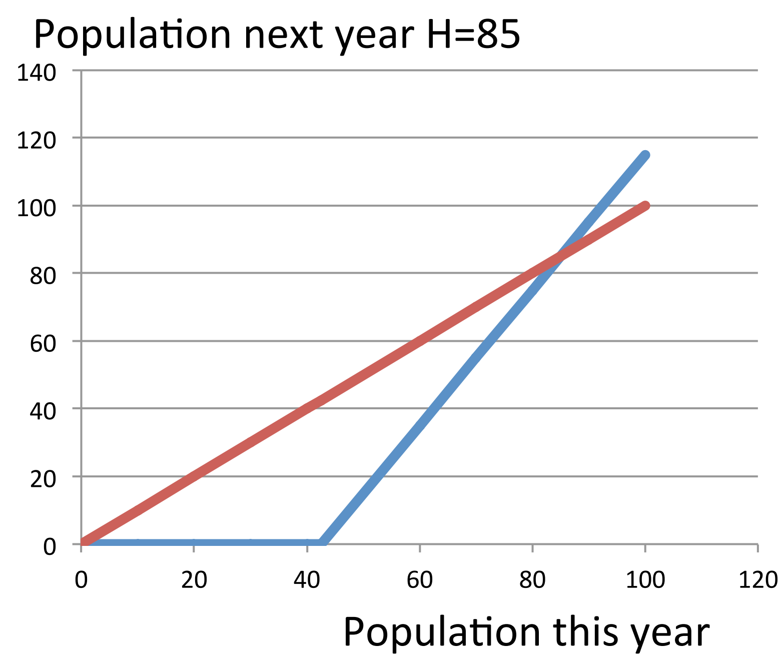




Exponential Growth with Harvesting

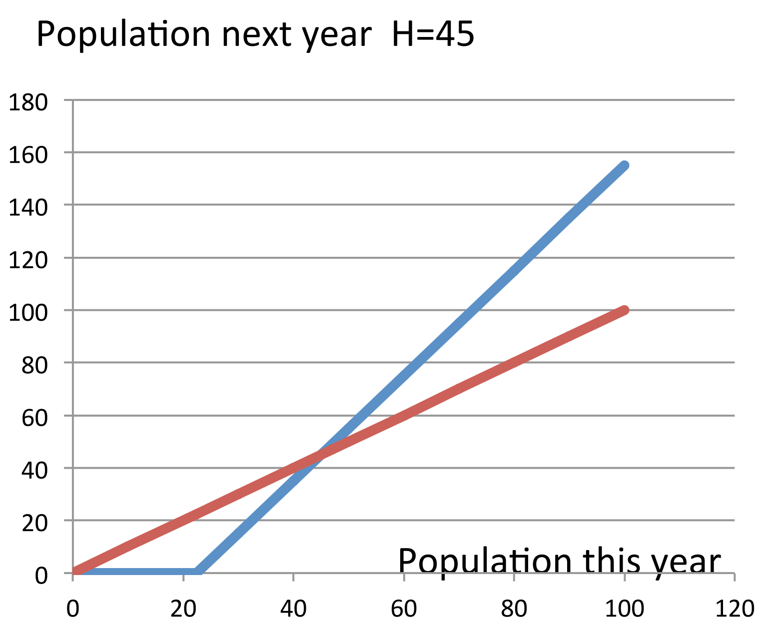
So, we learn that if we want to eliminate an invasive species, we either need to start early (when the population is very small) OR make the harvesting rate very large—both are politically difficult…but if you don't, then the model predicts the invasion will grow exponentially

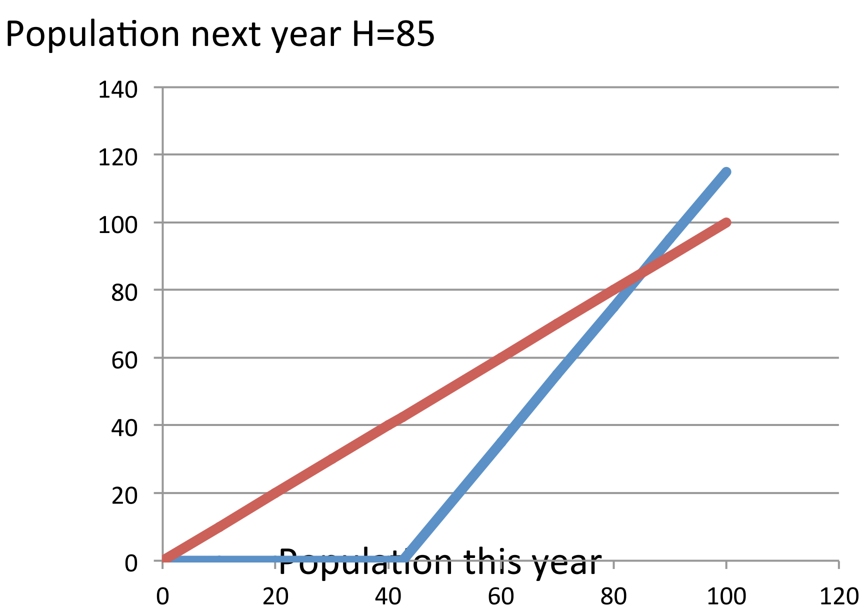




On the other hand…

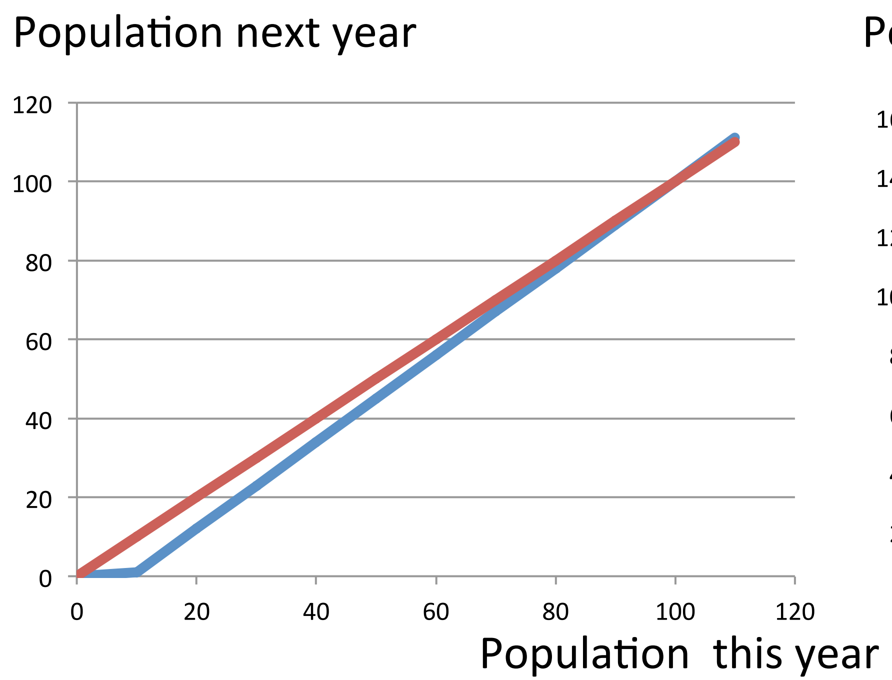
If we like having this species around and want to sustainably harvest (so make sure it does NOT go extinct), we need to make sure that our population is above or at the fixed point, so H can not be too large.

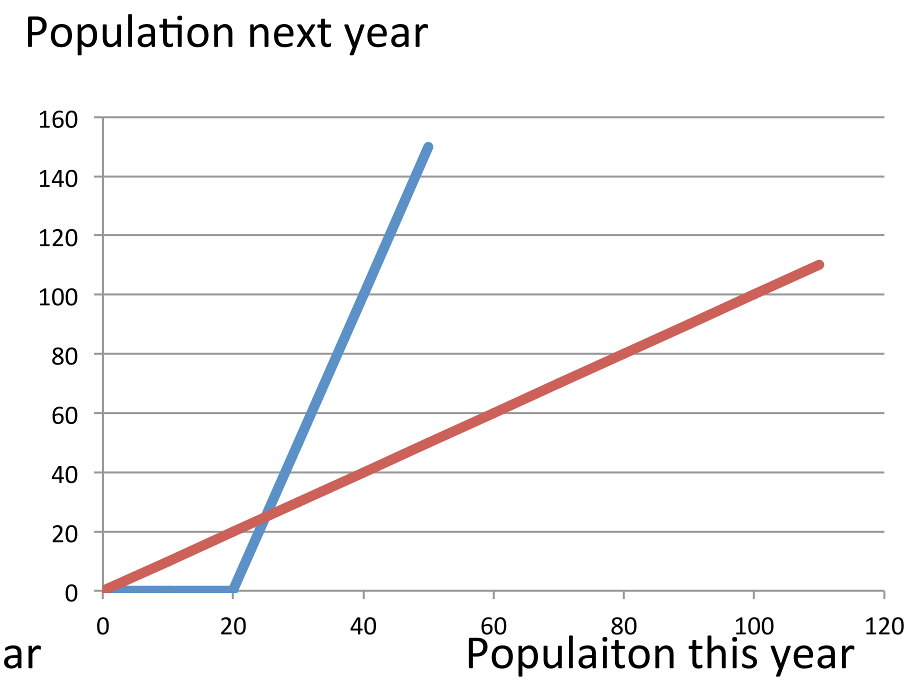




Quiz

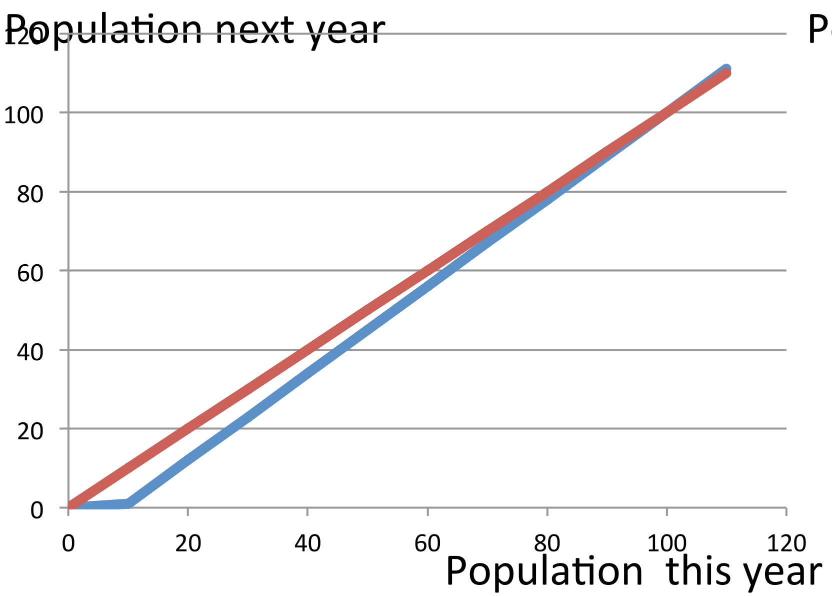
One graph below is mosquitos (that we murder with abandon) while the other is for rhinoceros (that we try to protect…) Which is which?

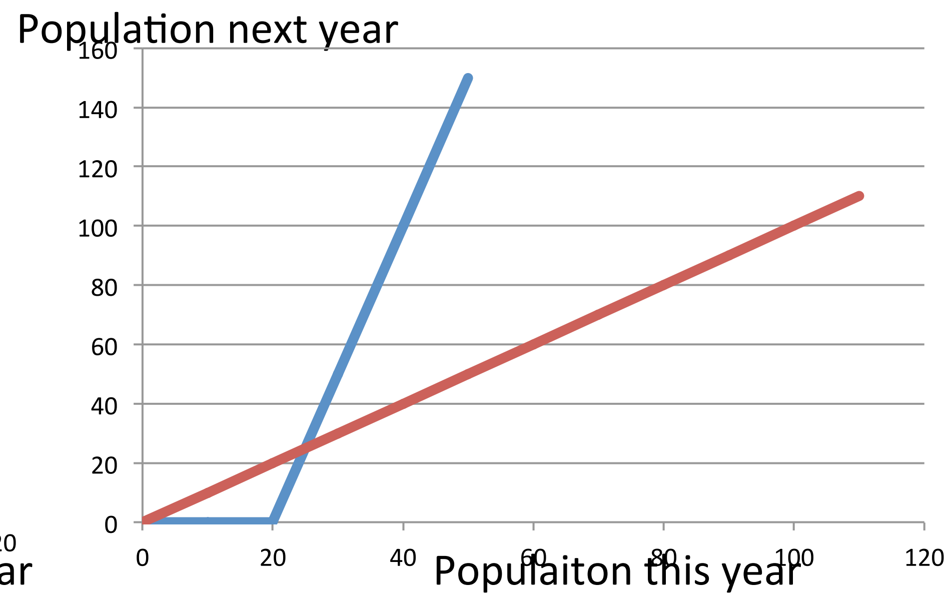




Lesson

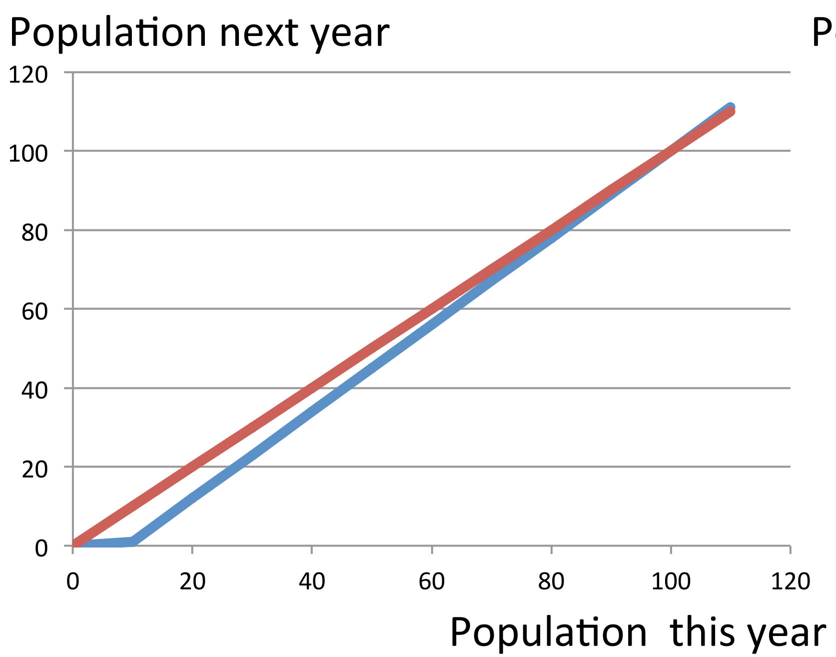
This is why rhinoceros are endangered and mosquitos are not…large k allows a relatively small population to be harvested without extinction, but k near 1 means we must have a small harvest or start with a large population.

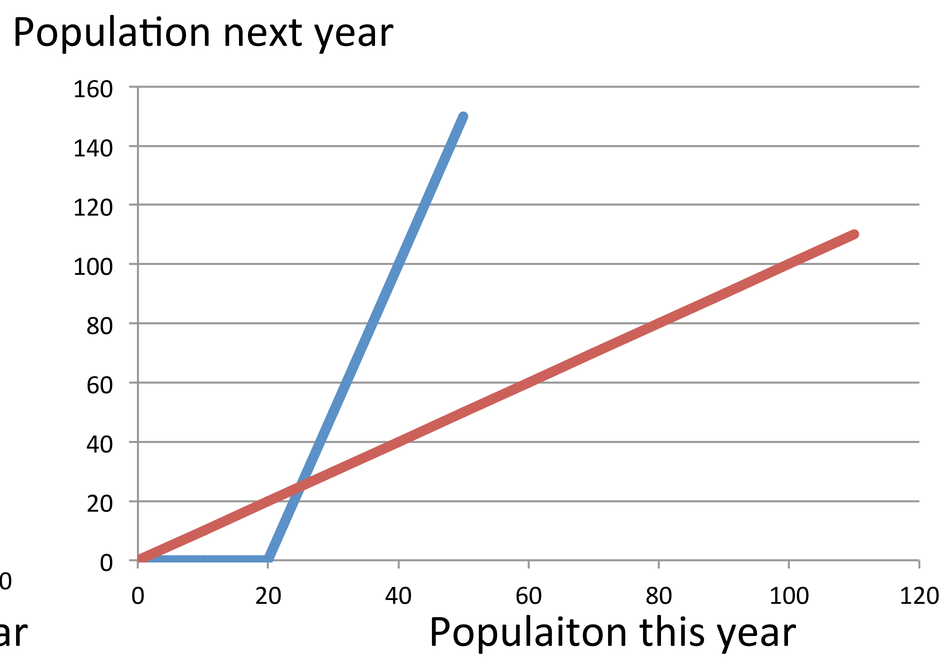




And, we can compute!

If we know the k, we can make the model and compute (ahead of time) how large a population we need in order to harvest at some partiucalar rate H.





A little Algebra

We can make this precise by computing the fixed point in general. For the model

$R(N+1)=k\,R(N) - H,$

The fixed point is where $R(N+1)=R(N)$, so

$R(N) = k\,R(N) - H,$

Or

$H = k\,R(N) - R(N) = R(N)\,(k-1),$

So the fixed point is at

Population = $H/(k-1)$.

Sustainable Harvest

If we have a population less than

$H/(k-1)$

then harvesting at rate H will cause extinction, if the population is larger than this number, harvesting at rate H can go on indefinitely.

(Or, if we have a population of more than

$H/(k-1)$

then harvesting rate H nwill not wipe out the pest species!)

Criticism of this model

There are several weaknesses of this model.

Most important is the lack of “stability”. In nature we see the population of the invasive species grow, but then level out and stay fixed. In this model, if we are at the fixed point, one extra rabbit will cause an explosion of the population, one fewer and the population dies out.

Keeping the population at the fixed point

Since a tiny change in the population away from the fixed point value will cause either a crash or an explosion, if this model is accurate, then we must "actively manage" the population to prevent a small change from making a huge, long-term difference…

But nature seems to take care of itself…populations can get big (lots of rabbits in my neighborhood), but not gigantic (I seldom step on one).

Discussion Next Week on Exponential Growth with Harvesting

BRING YOUR LAPTOP to discussion.

In discussion today and tomorrow you will read a comedy/tragedy of a case of “harvesting”.

You will work on the Excel page in discussion (that I will mail to you over the weekend) on Exponential growth and Exponential growth with harvesting models

And complete the work before class, handing in your work on Thurs. 5 Oct. in class (next week’s homework).

Suppose we give up on Harvesting…

If we give up harvesting and “let nature take its course”, eventually, even the most prolific of species will run out of food or space and the population will not become huge…

Can we model this situation?

Limited Growth

We start with the exponential model (first 3 assumptions) which work well for small populations, and add one more assumption

New Assumption 4: If the population ever becomes too large (that is larger than some specific size that depends on the environment and the species) then the population goes extinct (all the food gets eaten and extinction results).

Modify the model

Let’s let C denote “too large” a population, that is, if the population is ever C or larger, then next year, the population is zero.

There are lots of ways to modify our model to include assumption 4…but we want the easiest way possible. This is Occam’s Razor Or K.I.S.S.—keep it simple (unless simple doesn’t work).

To include the new assumption, we want to modify the exponential growth model

Rabbits next year = k x Rabbits this year

In a way which leaves the model unchanged when the population is small, but makes it crash if the population gets large…our choice

Rabbits next year=k x Rabbits this year

x (1 - (Rabbits this year)/ (Crash Pop.))

Or, in notation used before

$$R(N+1) = k\,R(N)\,(1 - (R(N)/C))$$

where C = “crash population”= the population where rabbits eat all the food and go extinct.

This model is a bit more complicated—but k and C are constants (that depend on the biology of the rabbits and the size of the environment) and it is quadratic in R(N).

So the graph of the population this year vs. the population next year is a quadratic (a “sad” parabola since the $R(N)^2$ term has – in front).

This is called the Logistic Model.

Example of logistic model and its predictions

Let’s set some values for k and C so we can look at some detailed predictions.

Let k=1.6 and C=4 (just as example…). Later we will look at different k values, but keep C=4 fixed. So R is in millions of rabbits or tons of rabbits.

So $R(N+1) = 1.6\, R(N)\, (1 – (R(N)/4))$

What does this predict…

If we know R(0) we could compute R(1) from this formula,

$$R(1) = 1.6\, R(0)\, (1 - (R(0)/4))$$

and then compute R(2) from R(1) and the formula…

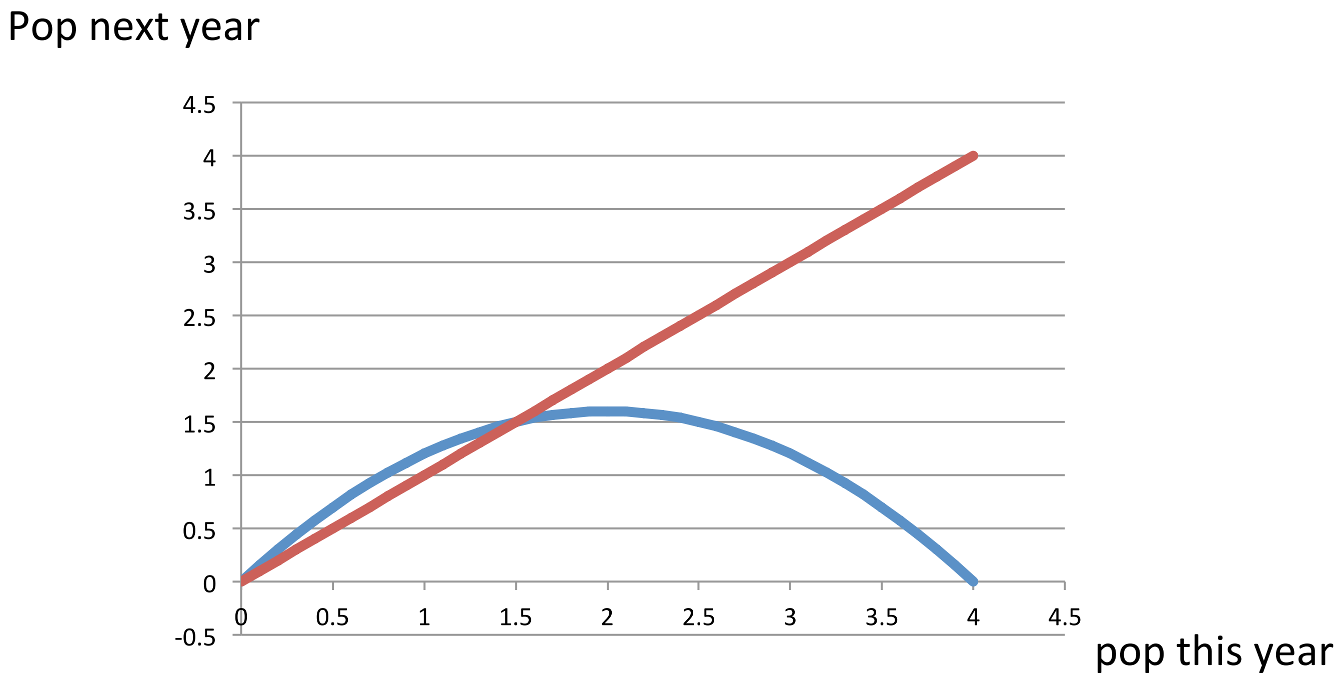
$$R(2) = 1.6\, R(1)\, (1 - (R(1)/4))$$

But this gets real tedious real fast…

Let's use the graph of population this year vs population next year to see what is predicted.

What the logistic model predicts

Looking at the graph of the population next year vs the population this year for C=4, k=1.6.



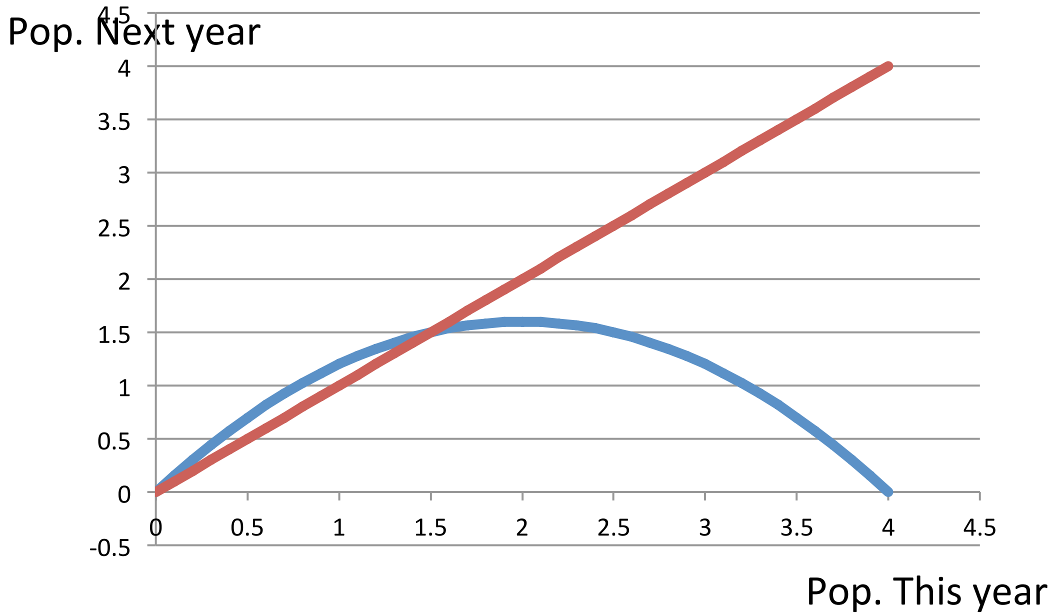
Blue logistic with k=1.6, C=4,

$$R(N+1)=1.6R(N)(1-R(N)/4)$$

Red is population this year=population next year

What can we tell from this picture?

When the population is very small, pop. next year is larger than pop. this year. And as the population grows, the amount of growth grows at least for slightly larger populations.

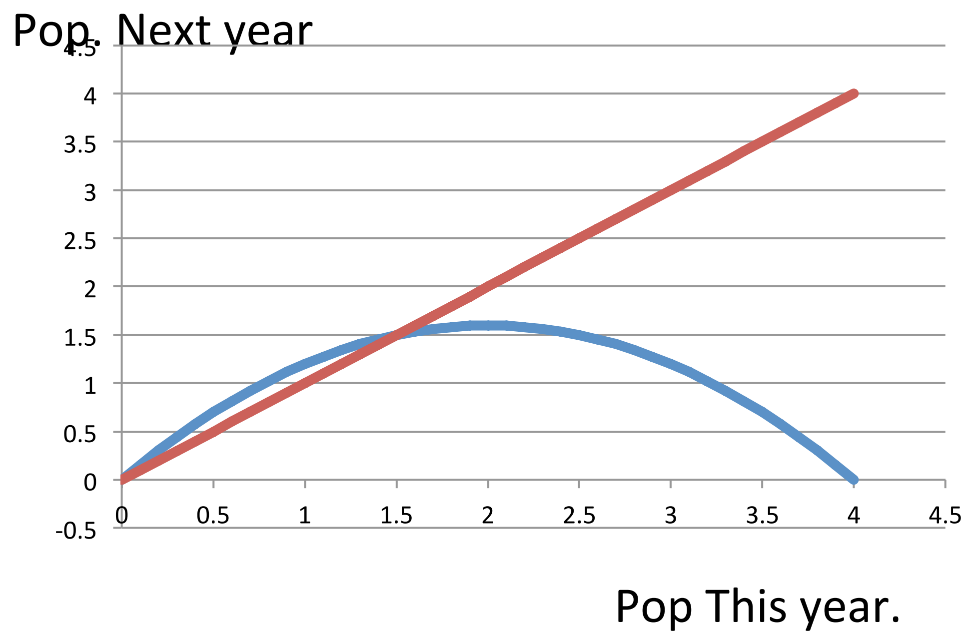


Blue logistic with k=1.6, C=4.

$$R(N+1)=1.6R(N)(1-R(N)/4)$$

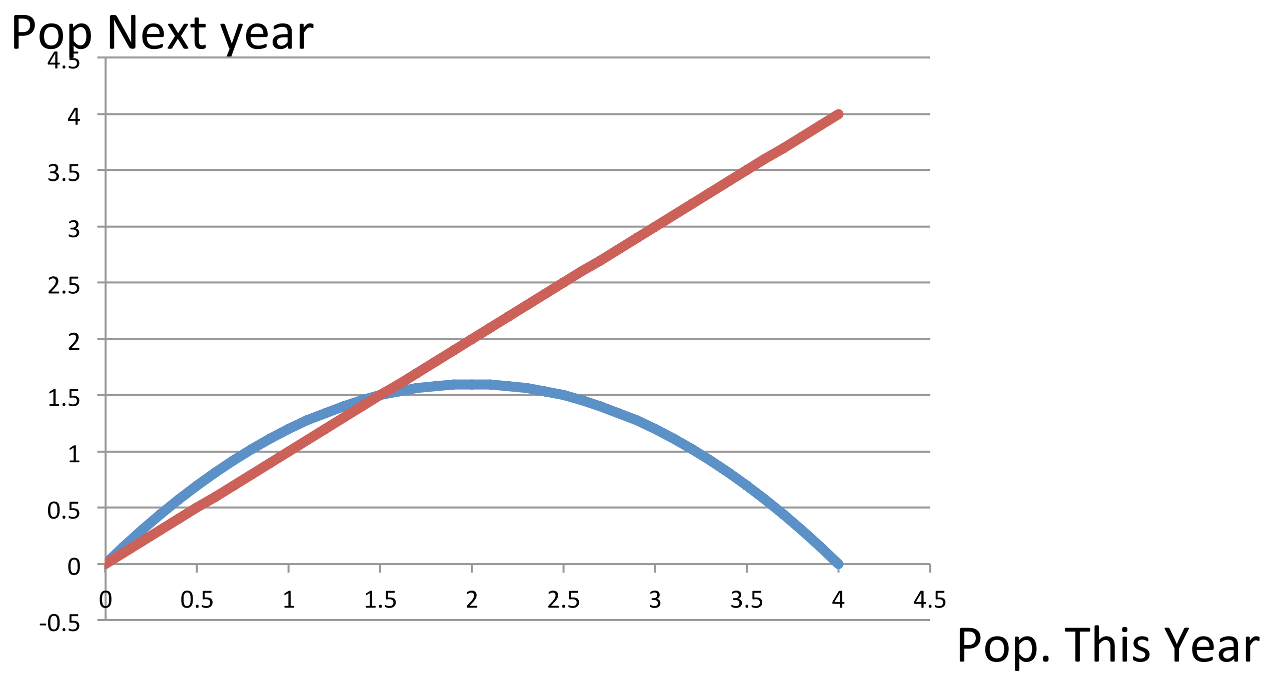
Red is population this year=population next year.

When the population this year is 4 (or higher), the population next year is 0 (extinct!). When the population is near 4 this year, the population next year is near 0 (crash)… population crashes down to the size where the population starts to grow again!



Blue logistic with k=1.6, C=4. Red is population this year=population next year.

In between (about population 1.5) the population next year equals the population this year (fixed point.)



Blue logistic with k=1.6, C=4. Red is population this year=population next year.

What about long-term predictions…if we start with a value of R(0), what happens to R(N) when t gets large? Say R(0)=0.3 (remember this could be thousands of rabbits, not 3/10's of a rabbit).

Then

R(1)=1.6R(0)(1-(R(0)/4))=1.6(0.3) (1-(0.3/4))=0.444

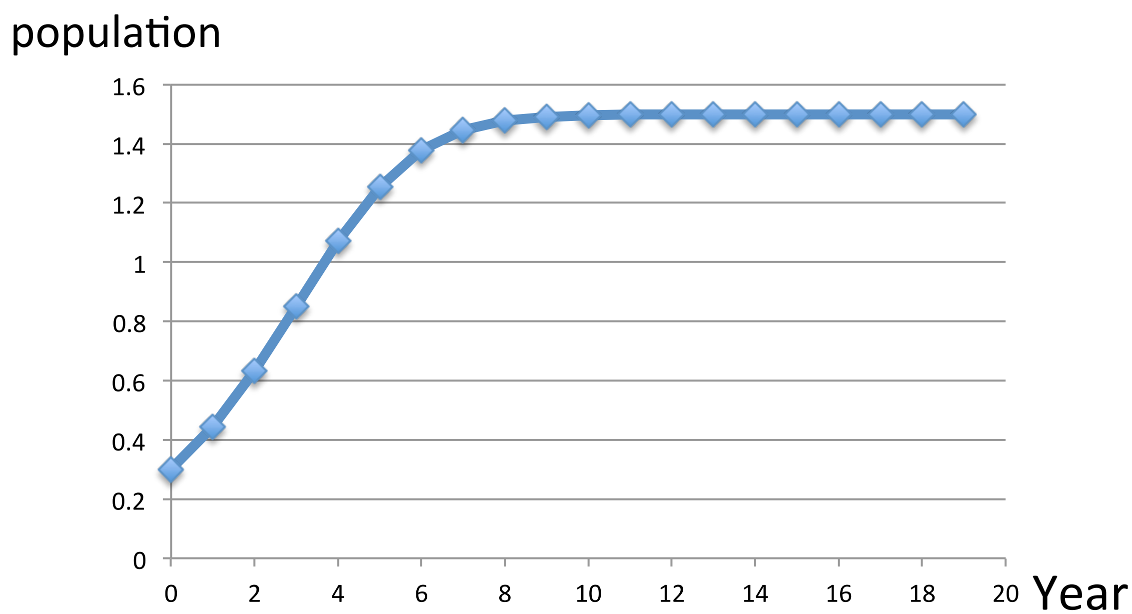
So

R(2)=1.6R(1)(1-(R(1)/4))=1.6(0.444)(1-(0.444/4))

=0.631

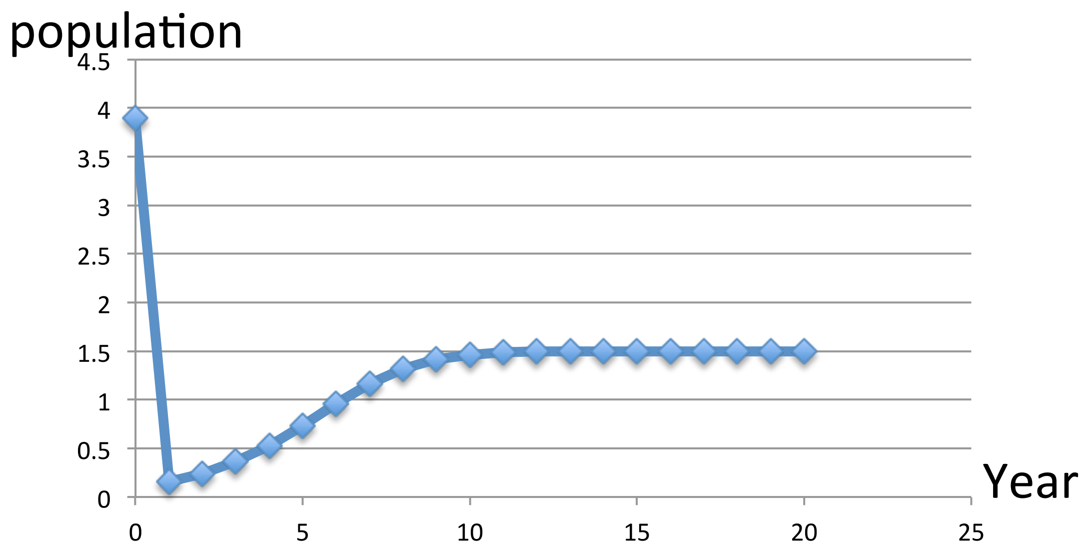
…this is tedious. Let Excel do it and look at the time series.

With k=1.6 and R(0)=0.3, the model predicts



That is, the population grows at first, but then levels off. That is often what we see with an invasive species. There is only a problem if the eventual population is larger than we would like.

What if we start with a different R(0)? Say we start with R(0) very large—say R(0)=3.9. (The Invasive species invades in force…) The model predicts



First there is a crash to near 0 population (3.9 is near the 4) but then the population grows as before and levels off at the same value, 1.5.

So with C=4, k=1.6, the model predicts that the invasive species population either starts small and grows quickly, then levels off

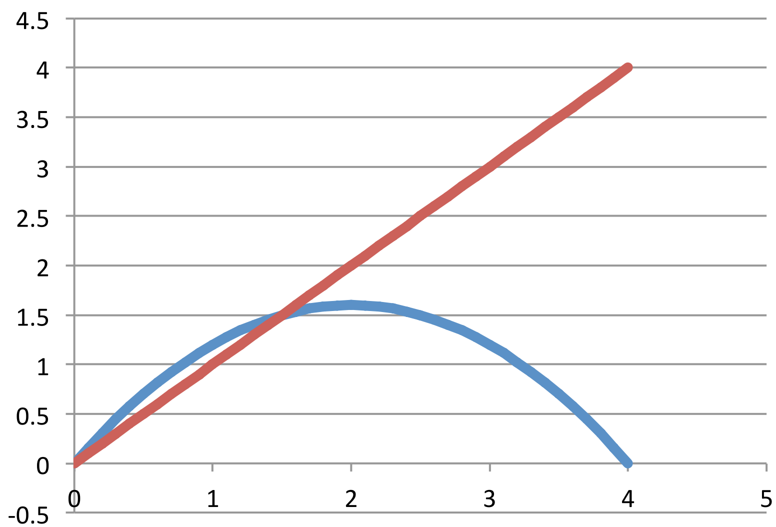
Or, starts large, crashes then grows and levels off …The long-term behavior seems to be the same for any initial R(0)

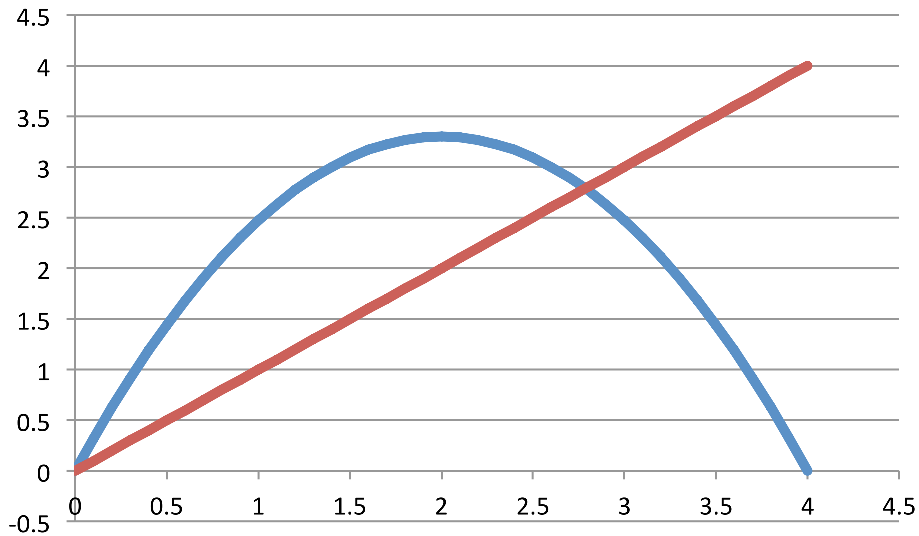
Do we get the same sort of prediction if we change the parameter values for k and C?

Let's explore (look at examples). Let's fix C=4 and try different values of k. We've done k=1.6. For this k value, any initial value R(0) between 0 and 4 leads to a "steady state" population.

This is nice. It was not a surprise to those who first studied this model…a simple model making a simple prediction…

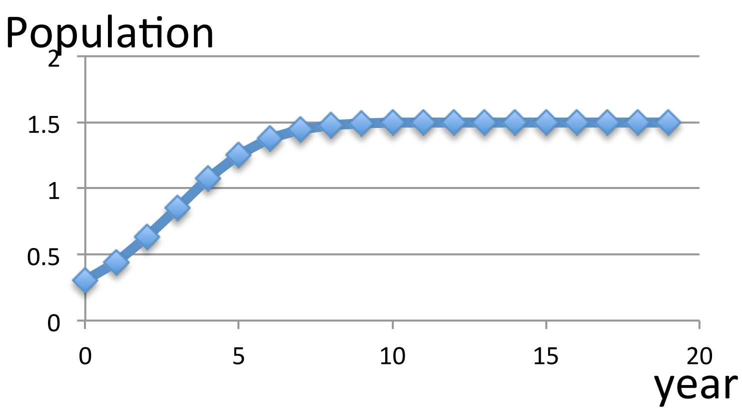
Try a larger k value. Turns out, when k gets bigger we start to see different types of predictions. For example, when k=3.3



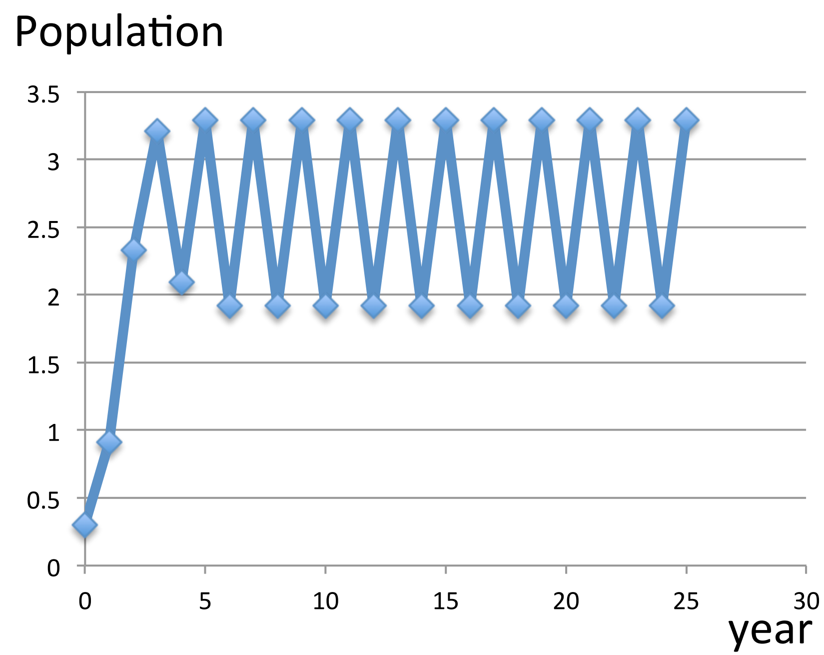


Left is k=1.6, right is k=3.3. Note that for k=3.3 the population grows more quickly when small, but still "crashes" at 4.

What is the predicted population? Say with R(0)=0.3 as before? (The vertical and horizontal scales are the same on both graphs.)

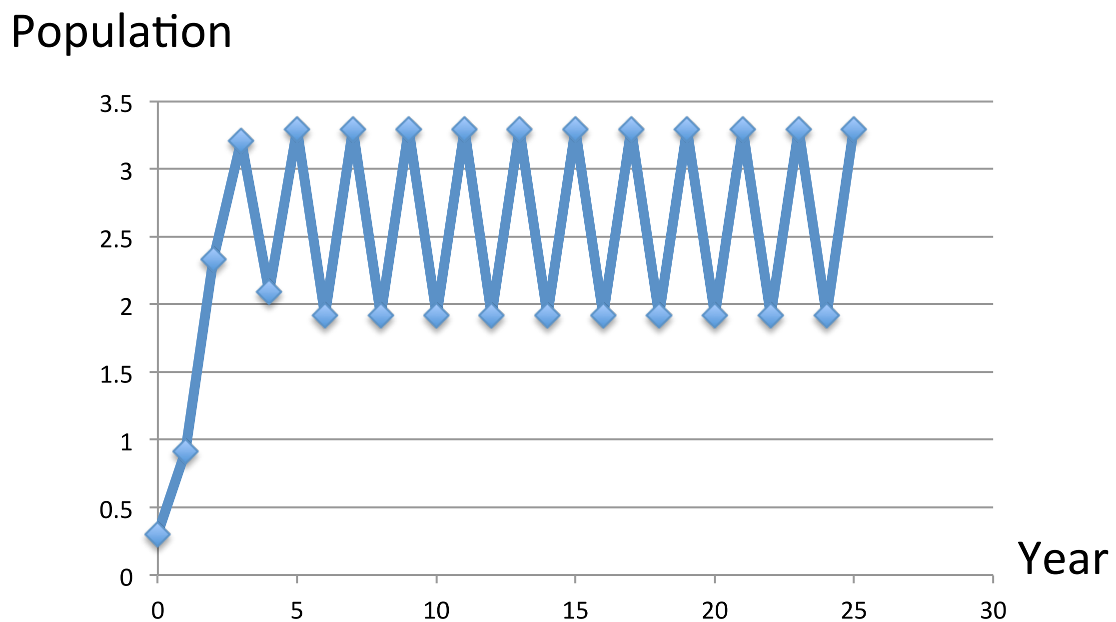


k=1.6, C=4, R(0)=0.3



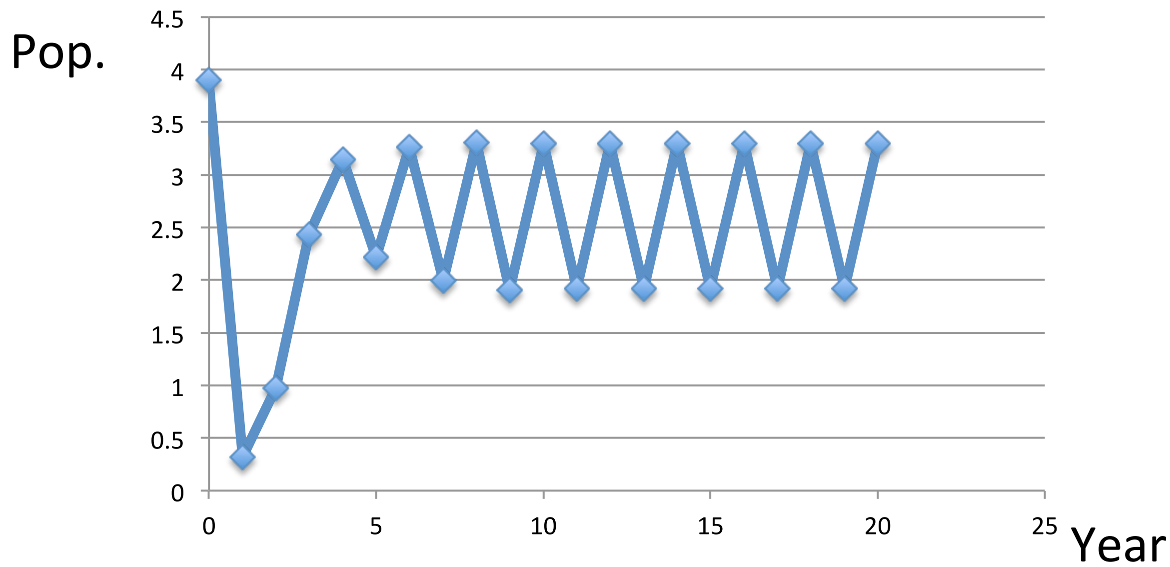
k=3.3, C=4, R(0)=0.3

Wow. This is very different!! When k=1.6 all populations level out, but when k=3.3 (same C and R(0) we get…



This is new! This shows the population growing at first, then oscillating back and forth “periodically”.

Let's try the same k=3.3 and C=4 but R(0)=3.9 (almost the crash value).

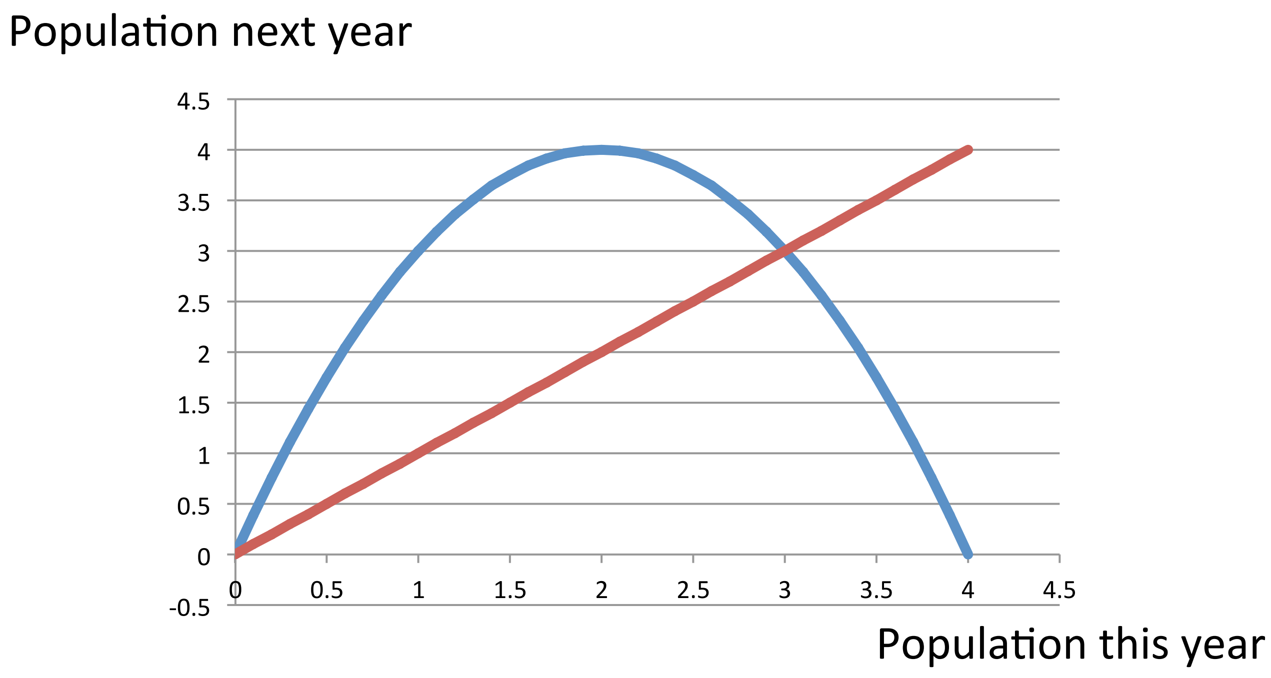


Again we see that the population crashes to near zero, then grows and falls into the oscillating periodic behavior.

So as the growth rate constant increases, we not only see faster initial growth, we see different possible long-term behaviors.

What happens if k gets even bigger?

Let’s try k=4 (same C=4)…The “one year in the future” graph doesn’t look that much different than k=3.3, C=4…



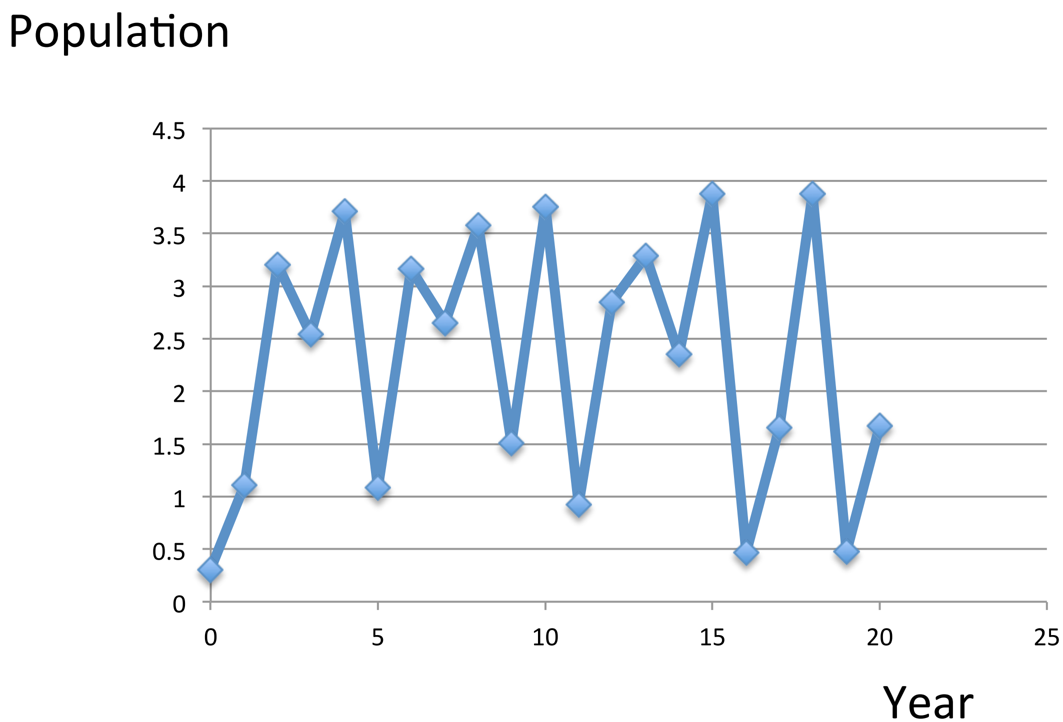
Try the same initial data $R(0)=0.3$ and see what is predicted in the future...

Will it

Tend to a constant steady population?

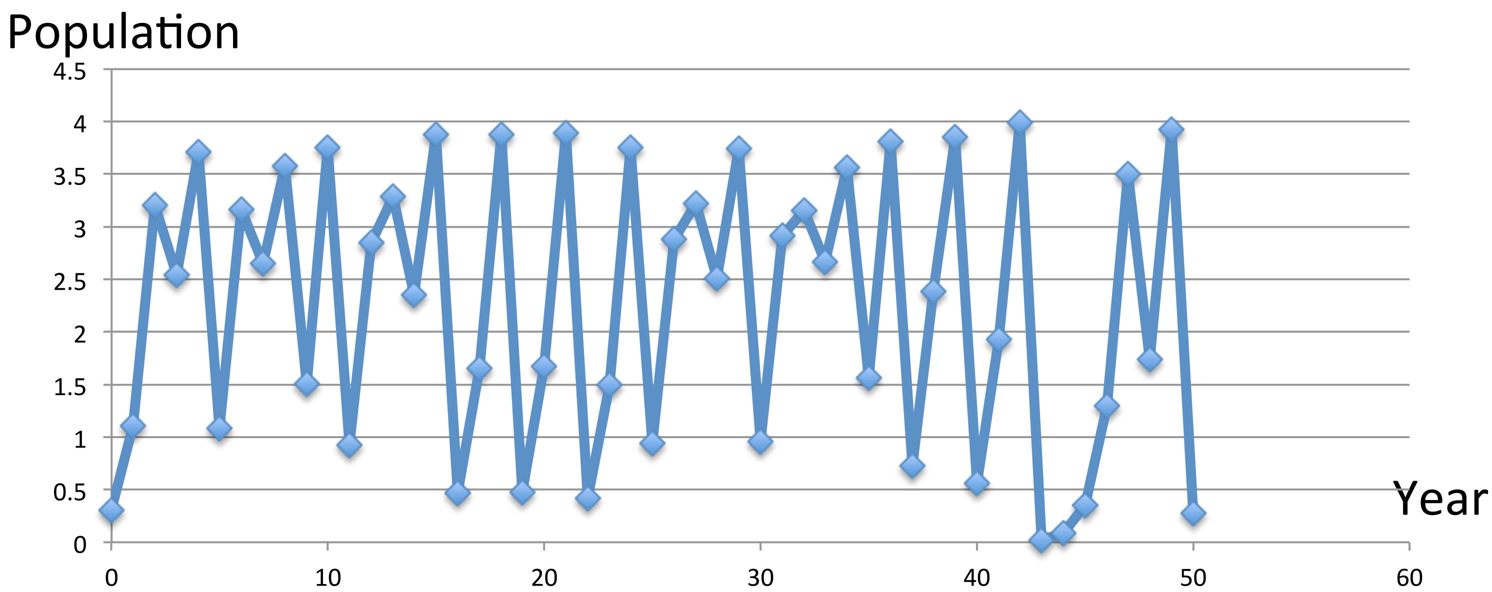
Oscillate back and forth periodically??

Try the same initial data R(0)=0.3 and see what is predicted in the future...Nope...



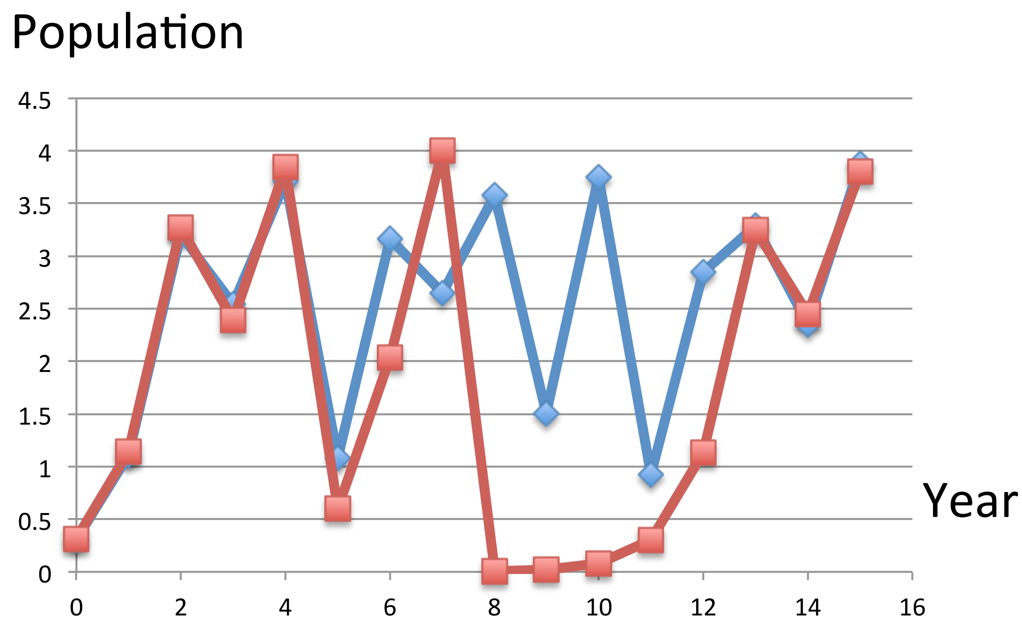
This is new! Maybe it just hasn't "settled down" yet. Let's look at this prediction farther into the future.

Same k=4, C=4, R(0)=0.3, but time up to 50



Has not settled down! Population repeatedly grows quickly, then crashes, but never exactly the same values…This is very strange behavior!

Even stranger behavior...Same k=4 and C=4 but two different initial populations R(0)=0.3 and R(0)=0.31



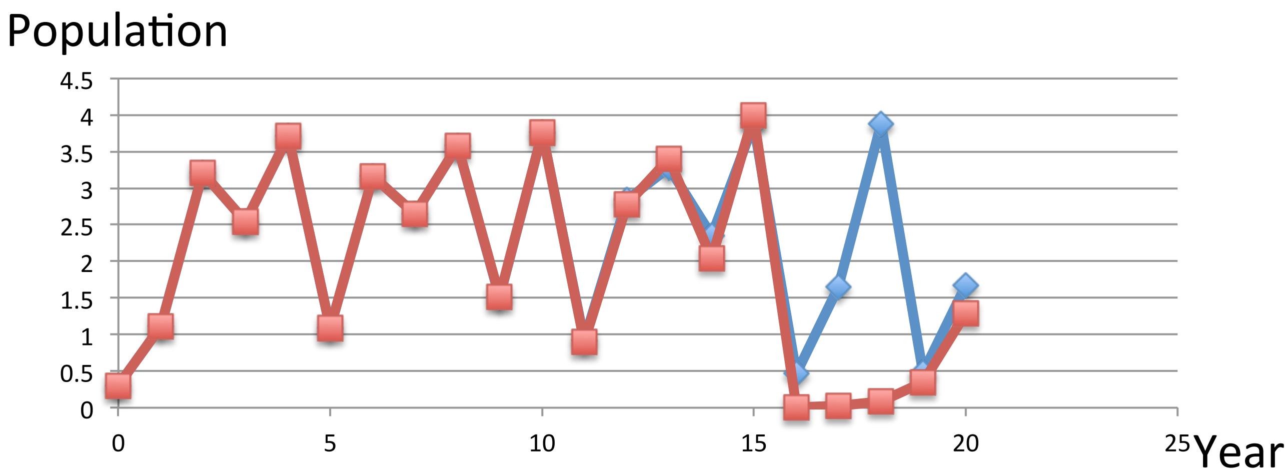
Orange is R(0)=0.31, blue is R(0)=0.3. The predicted populations are very close up to year 6, but then the separate dramatically.

We have seen this before!!

Remember that a small error leads to quickly to a big difference in prediction for the Exponential Growth Model. There it didn't matter much because we were predicting a population explosion in any case…

For the Logistic model there is still exponential growth when the population is small and this means any error in initial population will grow exponentially!

Even a tiny change in the value of R(0) leads, relatively quickly to a big change in prediction.



Blue R(0)=0.3, Orange R(0)=0.30001. After 16 years a slight difference in prediction, after 20 years a huge difference in prediction!

What have we seen??

When k=4 (and C=4) the logistic model makes very strange predictions…

1. The long-term behavior is wild, irregular, jumpy, “Chaotic”…
2. Even a tiny change in initial population R(0) leads to a radically different population prediction in relatively short time, or, “errors” or changes grow exponentially.

These are the hallmarks of a “Chaotic” system—the “New Science” of “Chaos Theory”

Pessimist says

For k=4 (and C=4) this model is useless…the prediction oscillates wildly AND any tiny error in the initial value R(0) (inevitable in the real world) leads to a wildly different prediction very quickly.

Optimist says

This is AMAZING! The simple Logistic model with k=4 and C=4 predicts really wild long-term behavior of the population AND a tiny change in initial value R(0) leads quickly to a very different prediction!

The real world is like that—systems like the weather and the stock market have wild behavior and it seems in life a small change can lead to a huge difference very quickly AND the simple Logistic model predictions behave like that!!

So complicated behavior might be happening for simple reasons! If the simple Logistic model can predict complicated behavior, maybe the complications we see in the world around us, at least sometimes, have simple explanations!!

There is hope that we can understand how things like the weather and the human brain work (even if we can not predict them exactly far in the future with any accuracy….)

Ask Answerable Questions

Also, this helps us to ask questions we can answer… for the Logistic with k=4, C=4, we can ask about how the population spreads out over time or what is the average number of years between population crashes…

(For the weather, we can't hope to predict if it will rain 1 year from today, but we can understand the trends in average temperature and rainfall…)

This is the science of “Chaos Theory”—It is a very optimistic view of the world. We should start studying complicated things by making simple models.

This is also unique in your study of Math—everything you have learned up until now was discovered or invented by a dead white European male over 300 years ago…But Chaos theory has its roots about 100 years ago and the chaos in the Logistic model was first studied in this way in 1976.

Not all the experts in Chaos Theory are dead, or white, or male!

Professor Devaney has the office down the hall from me and he is quite alive—his definition of "chaotic system" is the commonly used one.

Professor Nancy Kopell, a "MacArthur Genius" award winner (with office the other way down he hall from me) uses these ideas to study models of electrical signals in the brain.

NEXT

Probability and Statistics…

You probably come to these subjects with some baggage…

What is Probability? Why do we need it?