CS 112 – Introduction to Computing II

Wayne Snyder
Computer Science Department
Boston University

Today:

Efficiency of binary trees;

Balanced Trees

2-3 Trees

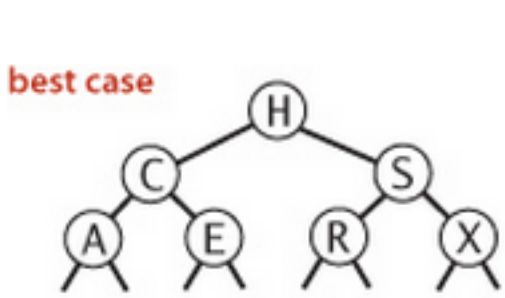
Next Time:

2-3 Trees continued

B-Trees and External Search

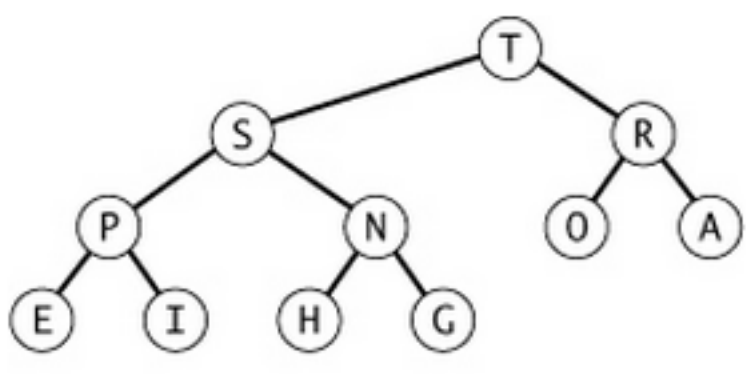
Efficiency of Binary Search Trees

So far, we have seen that the best case for a BST is a perfect triangle, and the worst case is a linked list:



Of course it may not be possible to get a

perfect triangle, but we can always create a tree in which the leaves are always within two levels of each other:



Best case: Θ(Log N)

Worst case: Θ(N)

What happens on average?

Efficiency of Binary Search Trees

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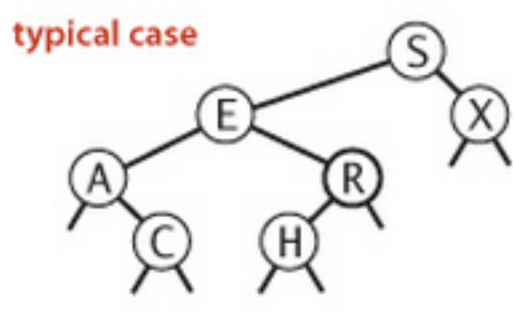
What happens on average? The scenario would be modeled on our experiments with average case for sorting:

- Create 1000 random BSTs for each size N = 1, 2, 3, 4, ... 100 (or similar parameters) by creating a random array of size N and then inserting each key into an initially-empty tree;
- Find the average cost of lookups in each tree (sum of cost of each node / N);
- This simulates a situation where a random BST is created, then we repeatedly lookup keys (we could alternately do a random series of inserts, lookups, and deletes on a single tree and see what happens – results are similar).



Cost of paths:
S: 1, E,X: 2, A,R: 3,
C,R: 4

Sum: 19

Average Cost: 19/7 = 2.71

Efficiency of Binary Search Trees

What happens on average? The scenario would be modeled on our experiments with average case for sorting:

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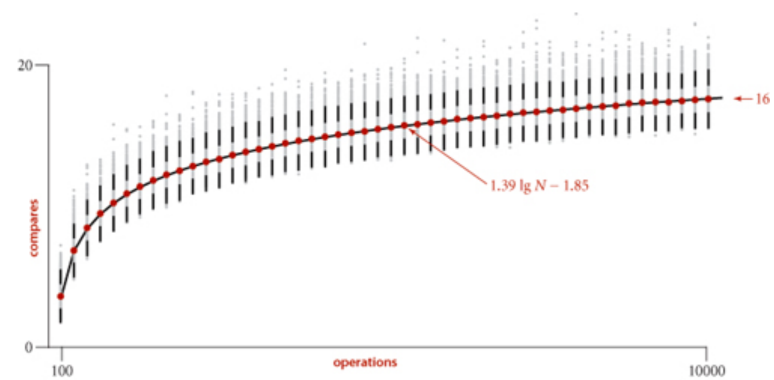
Result: Average case behavior of a BST is

$\Theta(\text{Log } N)$

You determined the constant C for the estimate ~(CN) in lab! Note that C was very small! This is an excellent result!



Average path length to a random node in a BST built from random keys

Balanced BSTs

Computer Science

The next question is always: **Can we do better?**

Specifically, can we find a way to eliminate the worst case trees, and get **Θ(Log N) for all operations?**

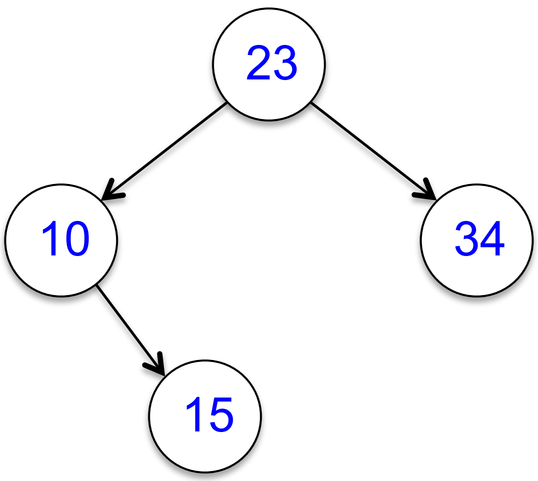
This amounts to the following problem: Can we restructure the tree during inserts and deletes to prevent imbalanced trees?

The answer, of course, is YES, and one solution to creating balanced trees is called 2-3 Trees....

2-3 Trees

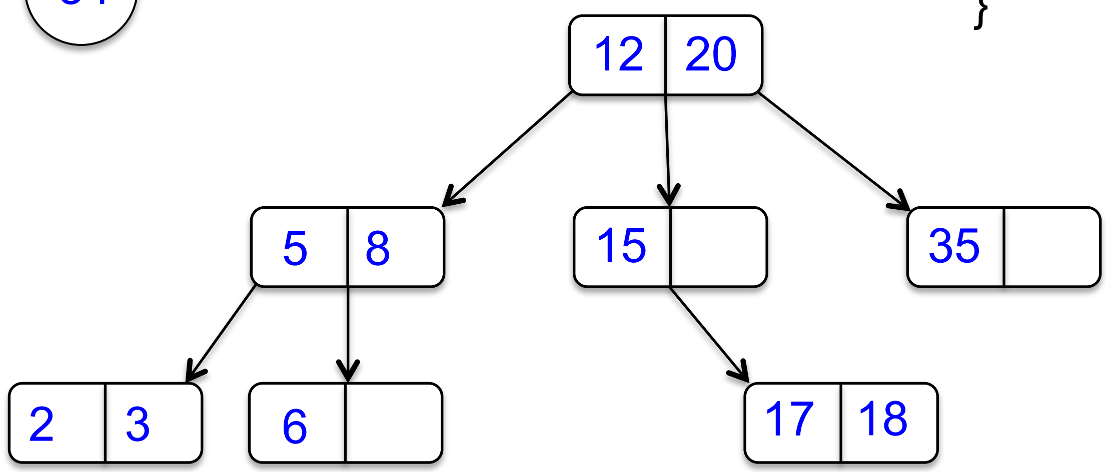
2-3 Trees generalize binary search trees by allowing “wider” nodes that can contain 1 or 2 keys, and 2 or 3 pointers:

Binary Search Tree:



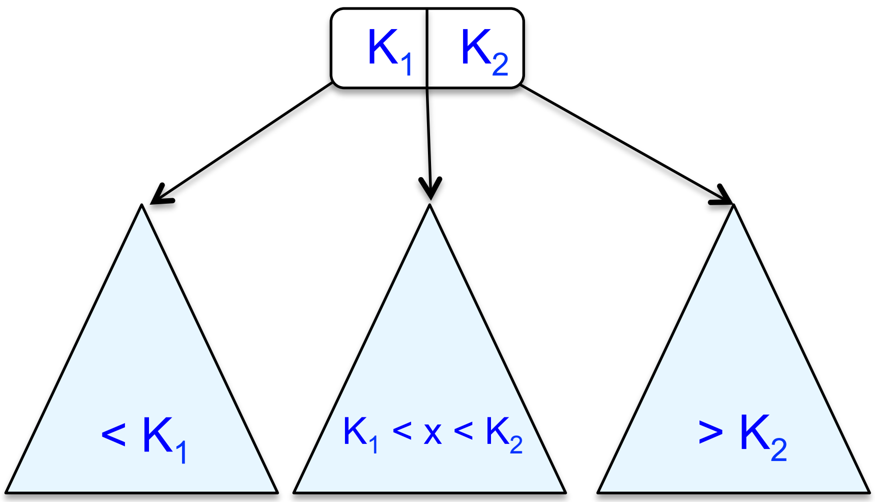
2-3 Tree:

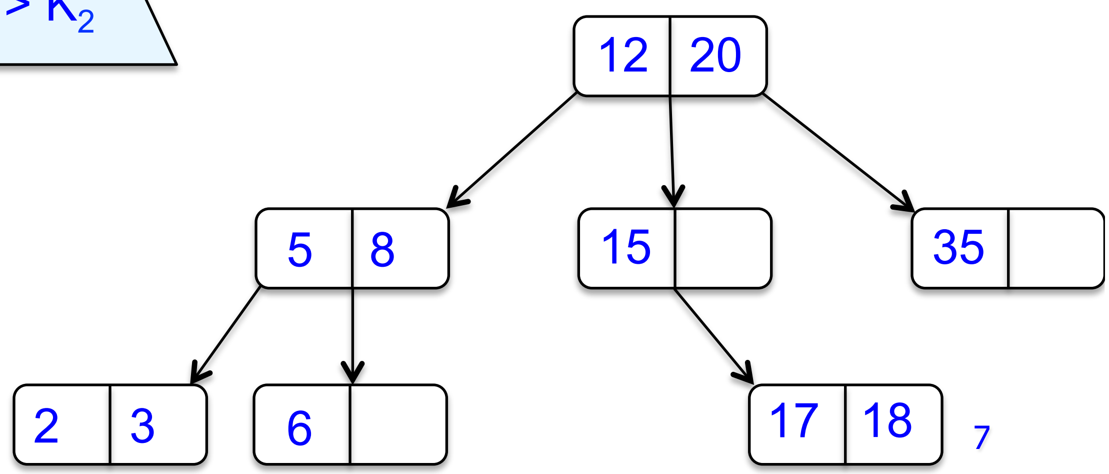
```
class Node {
    int K1, K2;
    Node left;
    Node mid;
    Node right;
}
```



2-3 Trees

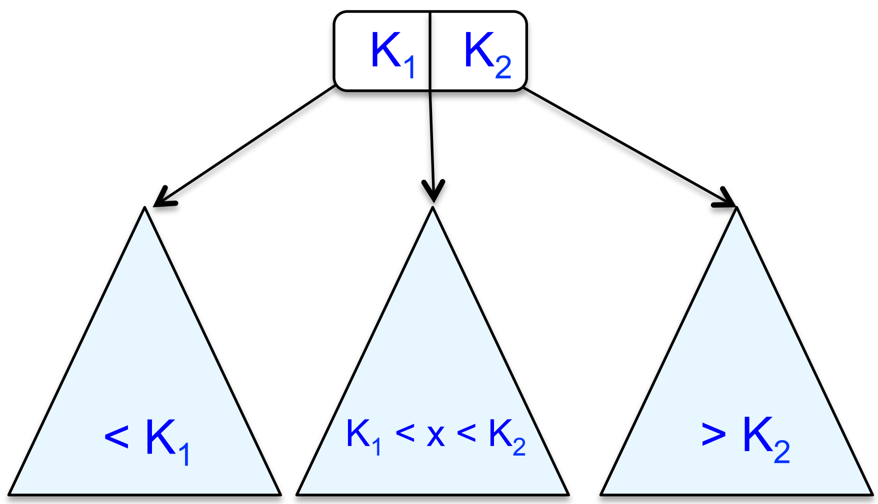
Generalizing the basic idea of binary search trees, we have "trinary search trees" where the two keys divide up the descendent nodes into three instead of two subtrees:

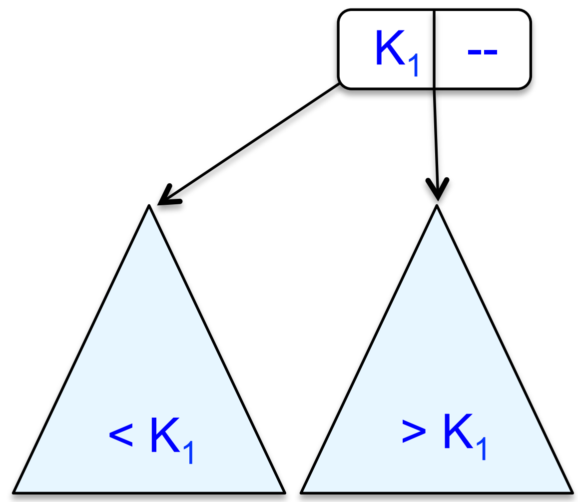


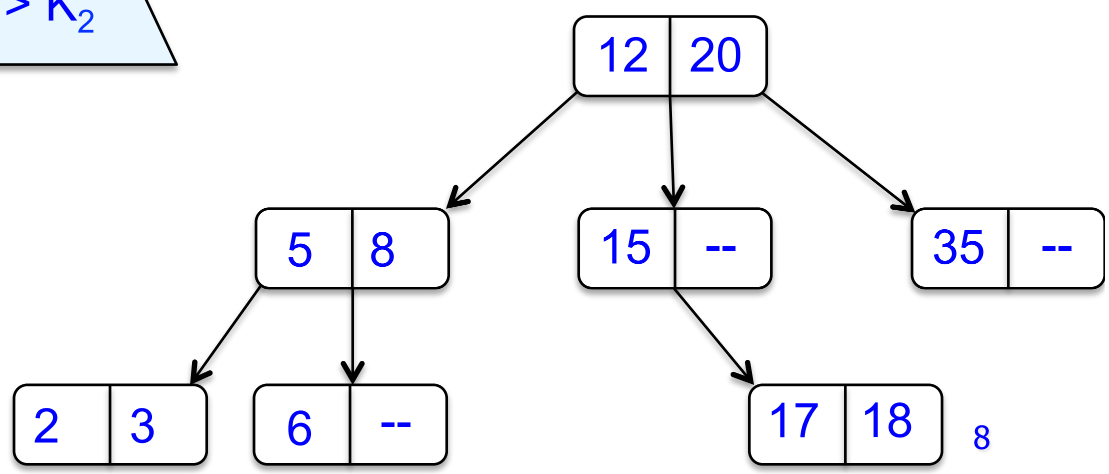


2-3 Trees

But we may consider normal BST nodes (1 key, 2 pointers) to be a special case, where the second key does not exist:

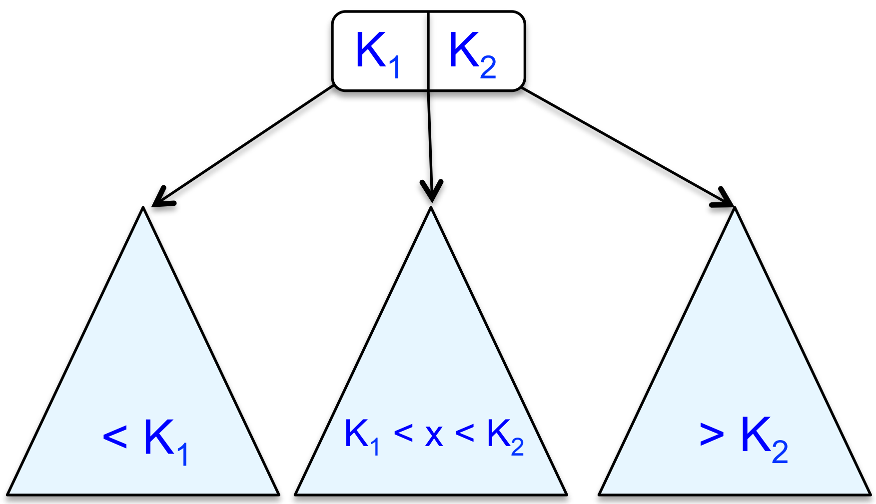


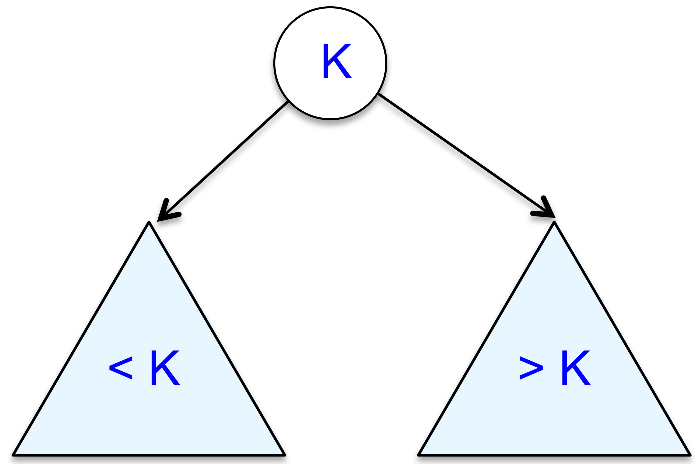


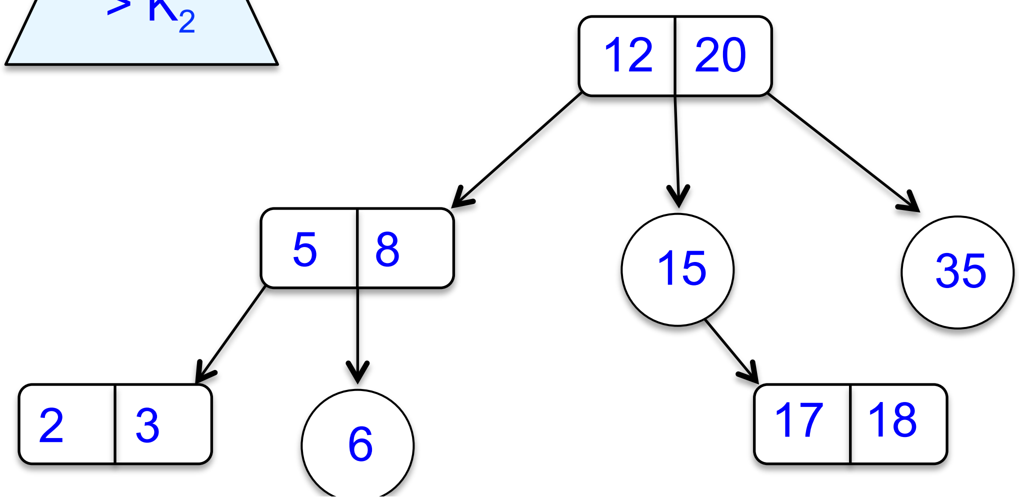


2-3 Trees

But we may consider normal BST nodes (1 key, 2 pointers) to be a special case, where the second key does not exist, and we will draw these as we would with normal BSTs:



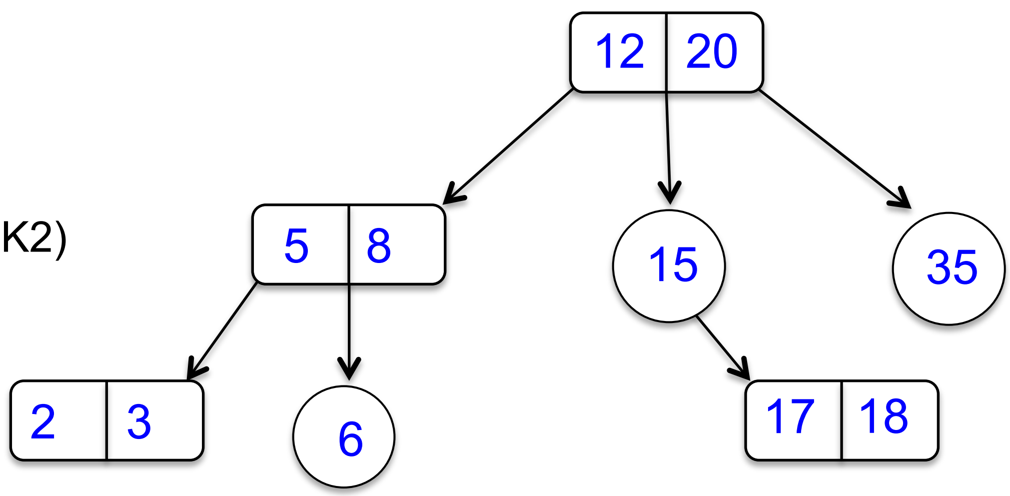




2-3 Trees

Searching such a tree is a simple generalization of search in BSTs: at each node you scan from the left through the two keys and figure out where the search key k might be:

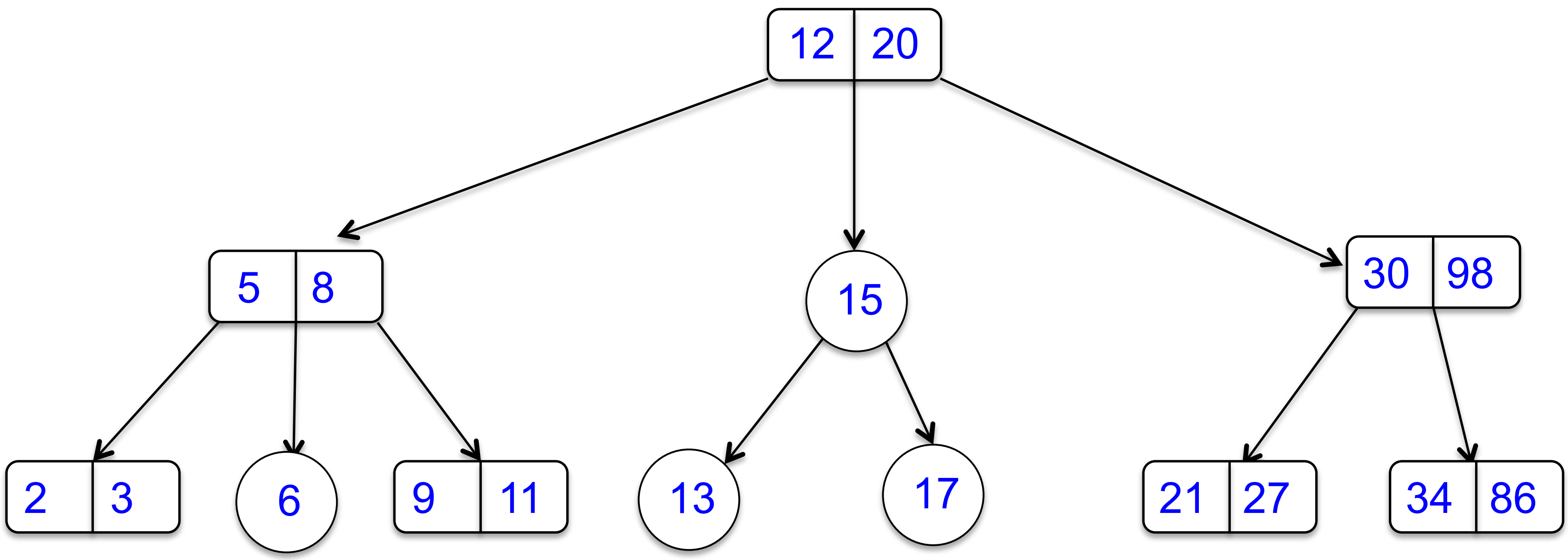
```
boolean member(int k, Node p) {
      if(p == null)
            return false;
      else if(k < p.K1)
            return find(k, p.left);
      else if(k == p.K1)
            return true;
      else if(p.K2 does not exist || k < p.K2)
            return find(k, p.mid);
      else if(k == K2)
            return true;
      else
            return find(k, p.right);
}
```



2-3 Trees

Insertion into a 2-3 tree is a little bit complicated, because we will want to maintain the trees in balanced form (perfect triangles):

A 2-3 tree is **balanced** if every path from the root to a leaf node has the same length; note that nodes may contain 2 keys and 3 pointers, or 1 key and 2 pointers:



2-3 Trees

Rules for inserting a new key into a 2-3 tree:

1. As with BSTs, you search for the key; if you find it, do nothing (don't insert duplicates); if you don't find it, then insert into the leaf node that you last looked in. If there is room, you are done.

Example: Let's insert a 12 into an empty tree; when you insert into an empty tree, you create a new node and insert into the K_1 slot:

12	--

2-3 Trees

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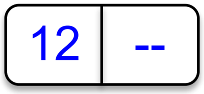
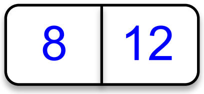


Now let's insert an 8, which can fit into the node if we move the 12 over:



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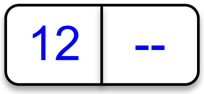

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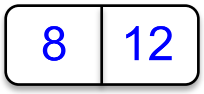
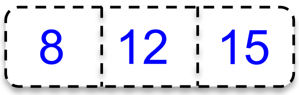


Next let's insert a 15, which expands the node into an *error node* containing too many keys:



2-3 Trees

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8	12	15

Immediately fix this error by transforming this node into a balanced three-node tree:

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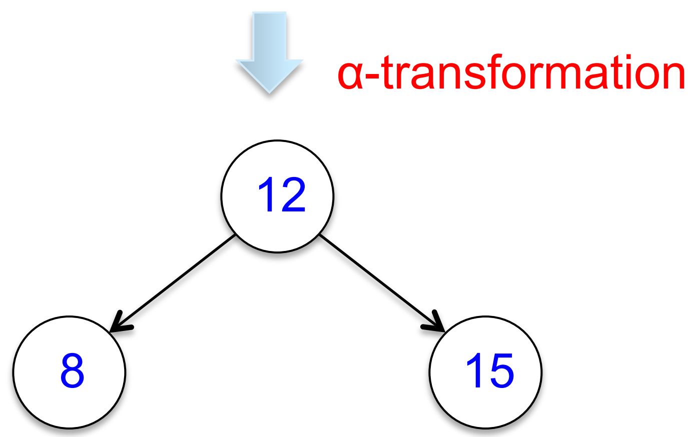
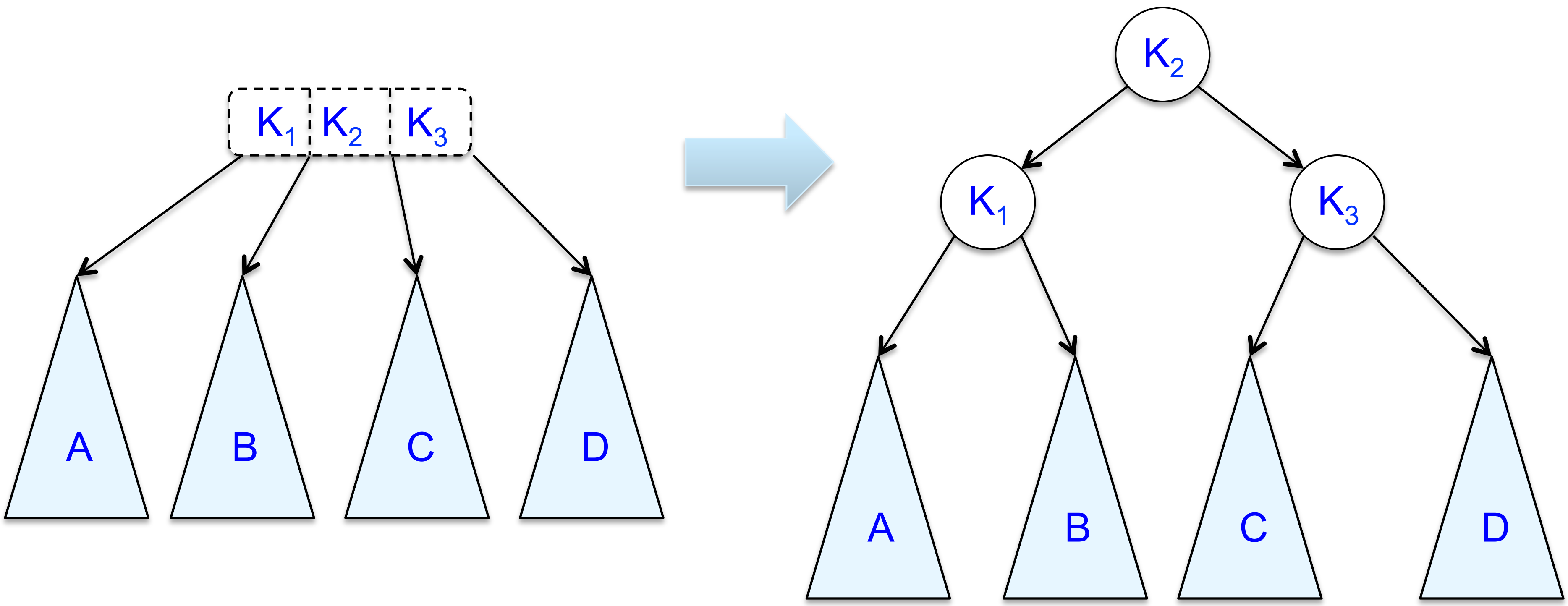


α-transformation:



The subtrees A – D may be null!

2-3 Trees

Rules for inserting a new key into a 2-3 tree:

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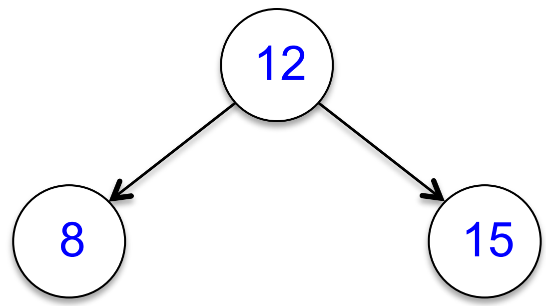
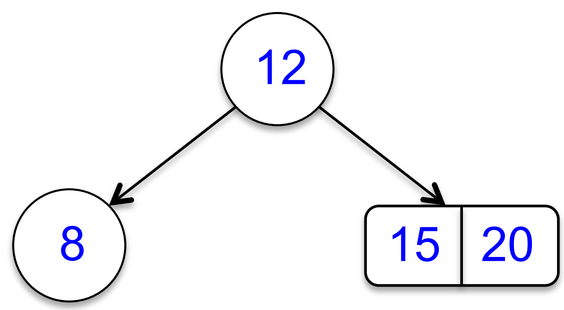


Next let's insert a 20, which expands the right-most leaf node:



2-3 Trees

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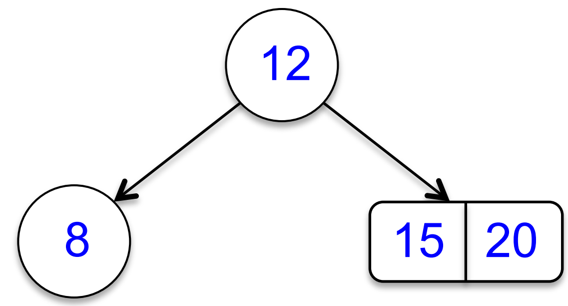
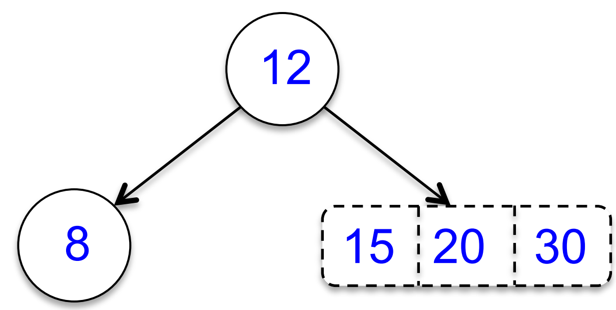


Then let's insert a 30, which creates another error node:



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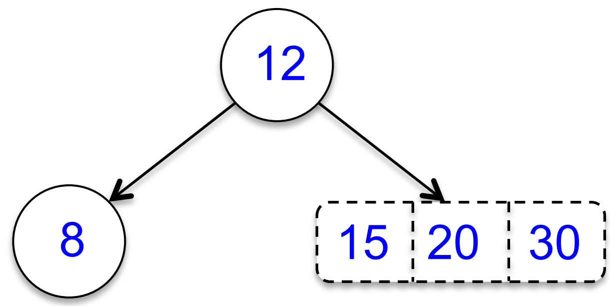

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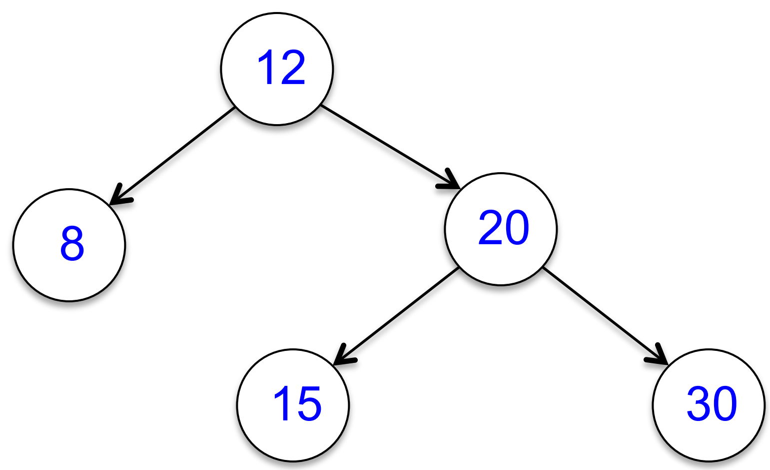

2-3 Trees

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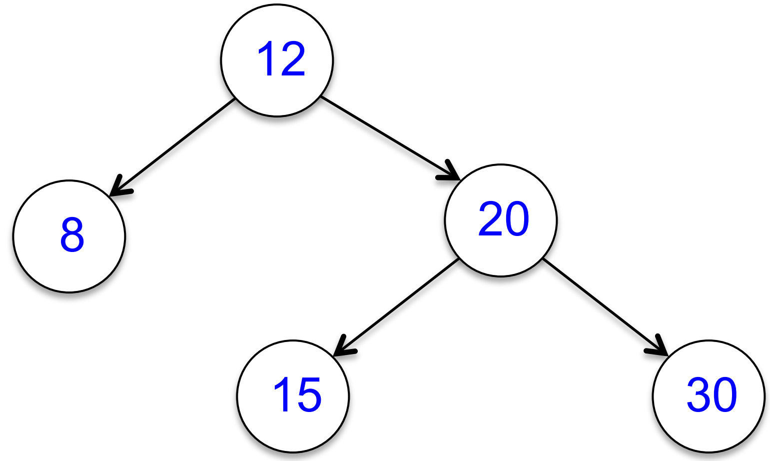
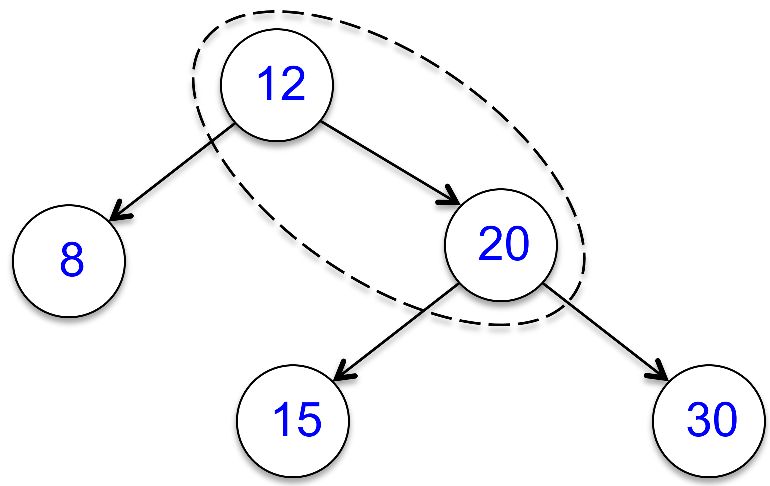


But this is imbalanced, so we will combine the root of the new subtree with the parent node:



2-3 Trees

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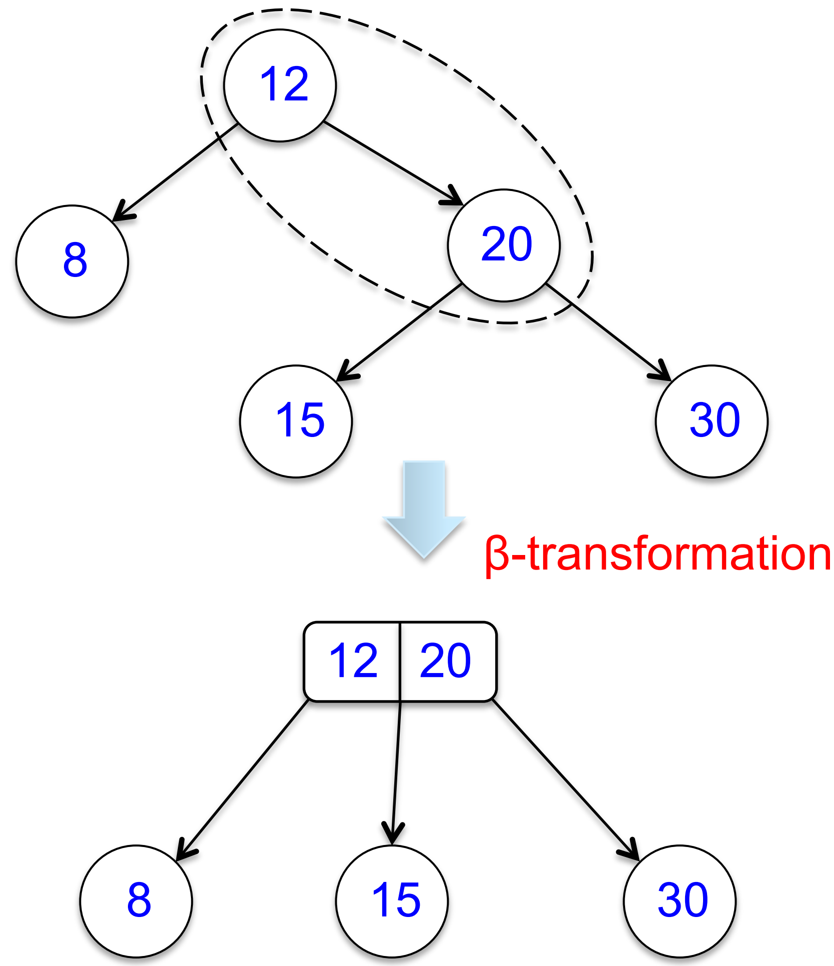
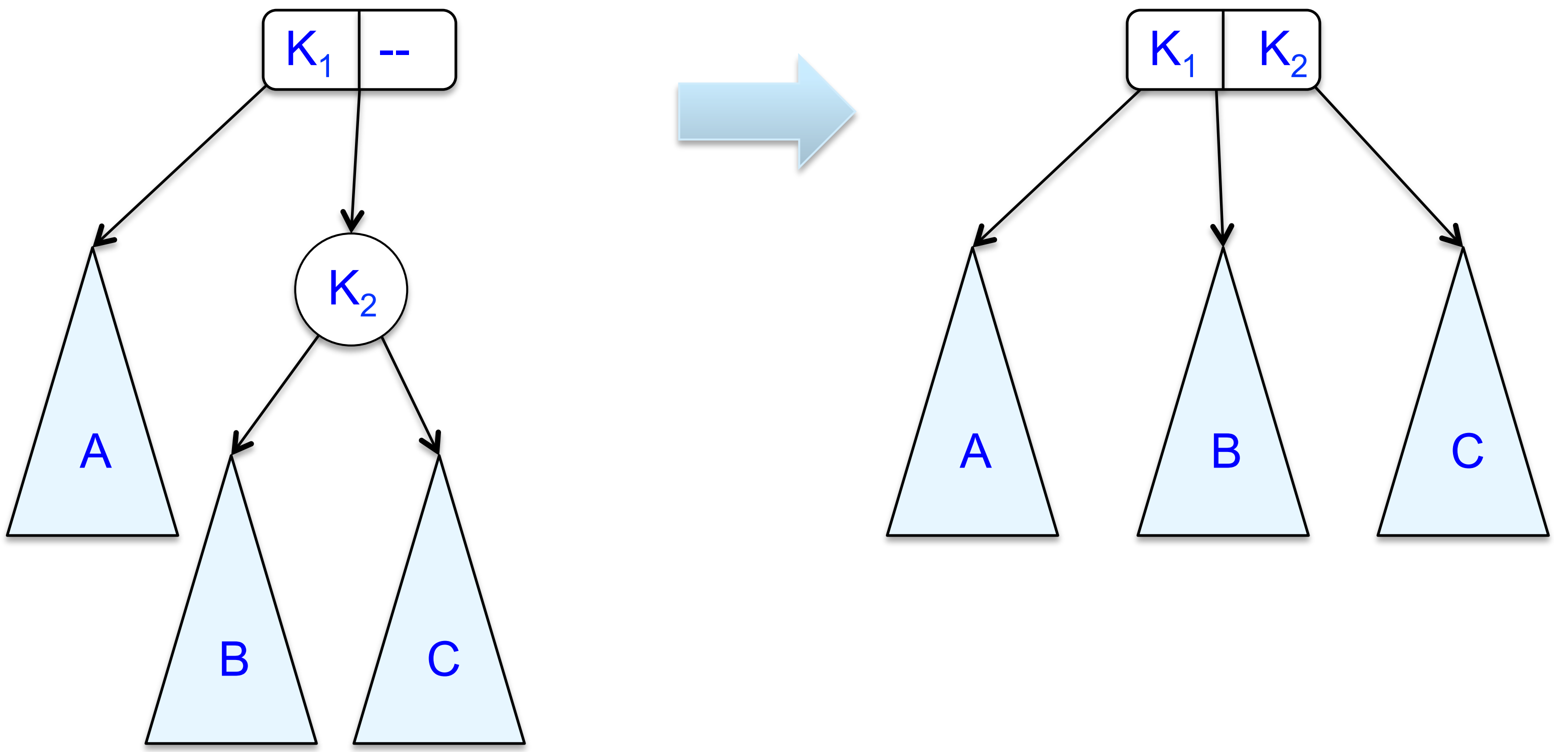


β-transformation(s): If the parent has only 1 key, then insert the root into the parent node and distribute the subtrees accordingly:



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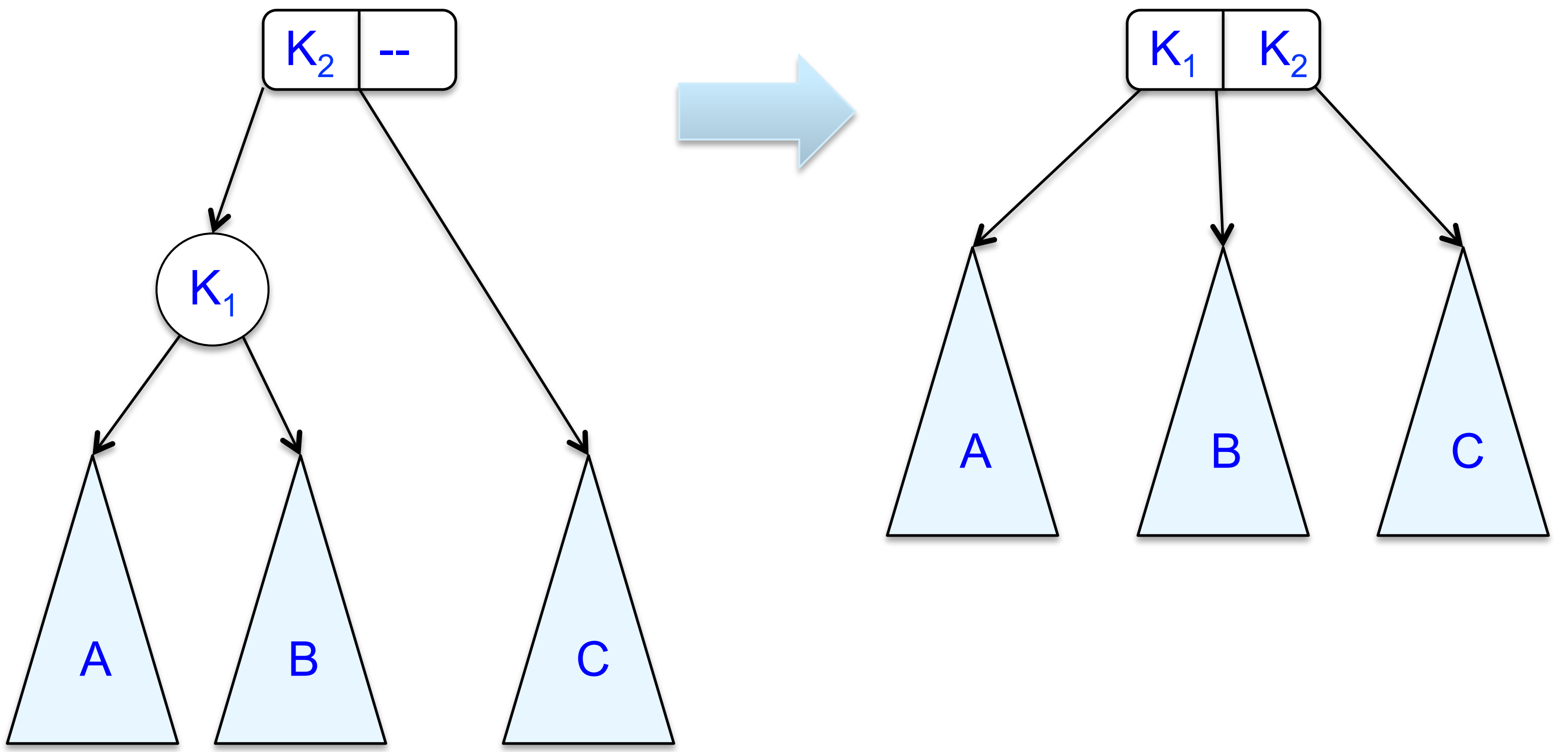
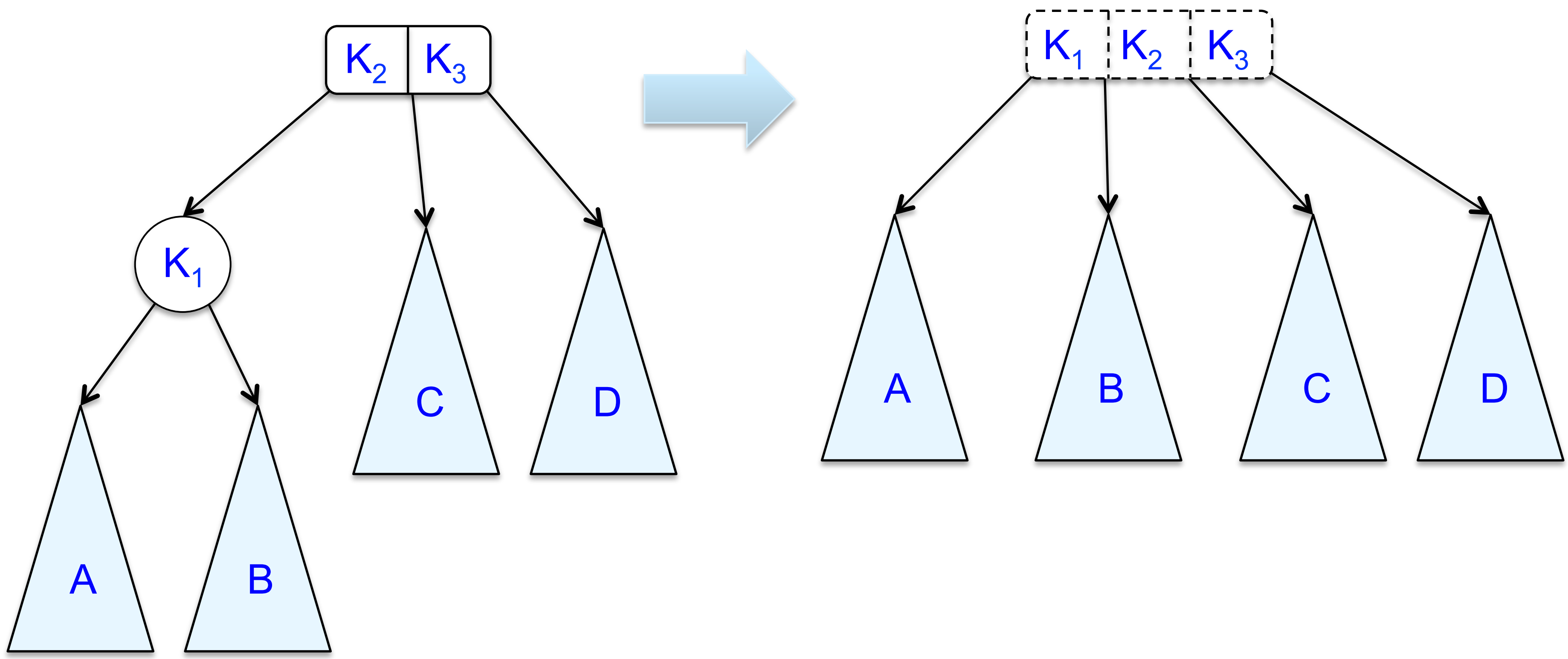


β-transformation(s): If the parent has 2 keys, then create an error node and repeat the α-transformation (you may have to continue apply α- and β-transformations up the tree):



2-3 Trees

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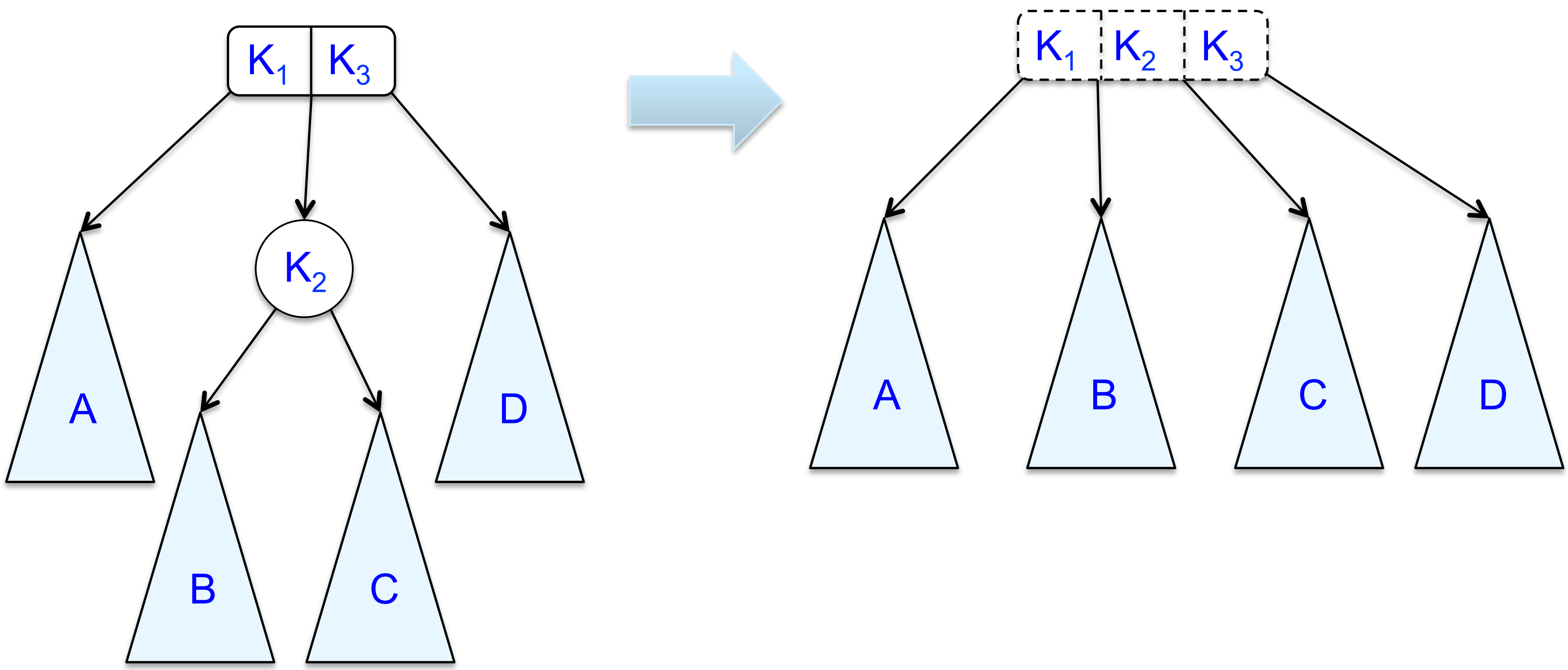
β-transformation(s): If the parent has 2 keys, then create an error node and go back to the α-transformation (you may have to continue apply α- and β-transformations up the tree):



2-3 Trees

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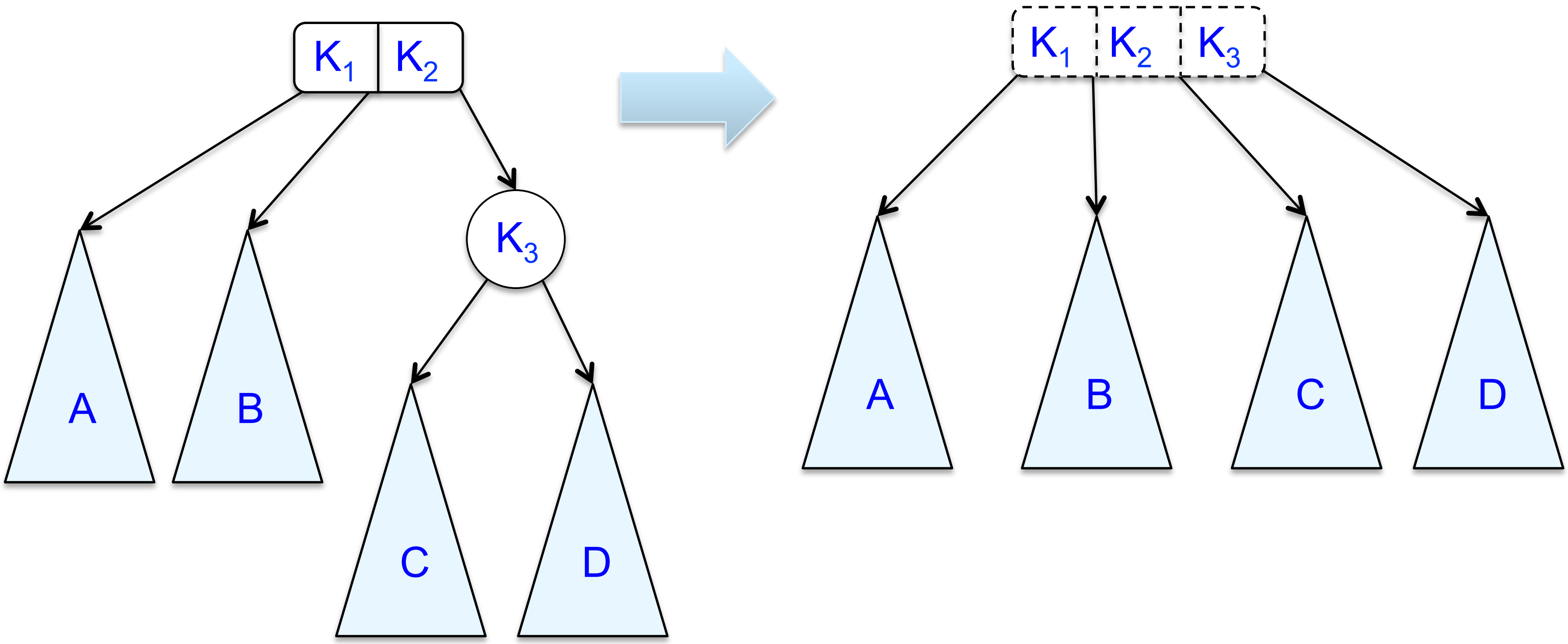

2-3 Trees

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Rules for inserting a new key into a 2-3 tree:

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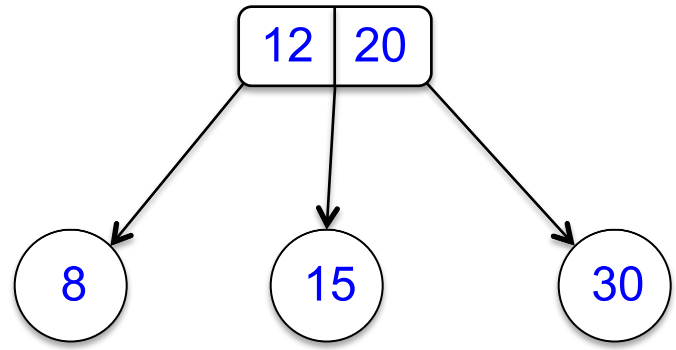
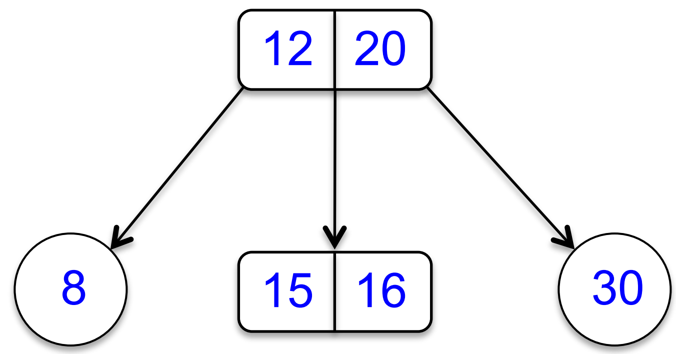
4. You may have to continue a series of α- and β-transformations moving up the path to the root, until a balanced tree with no error nodes is obtained.

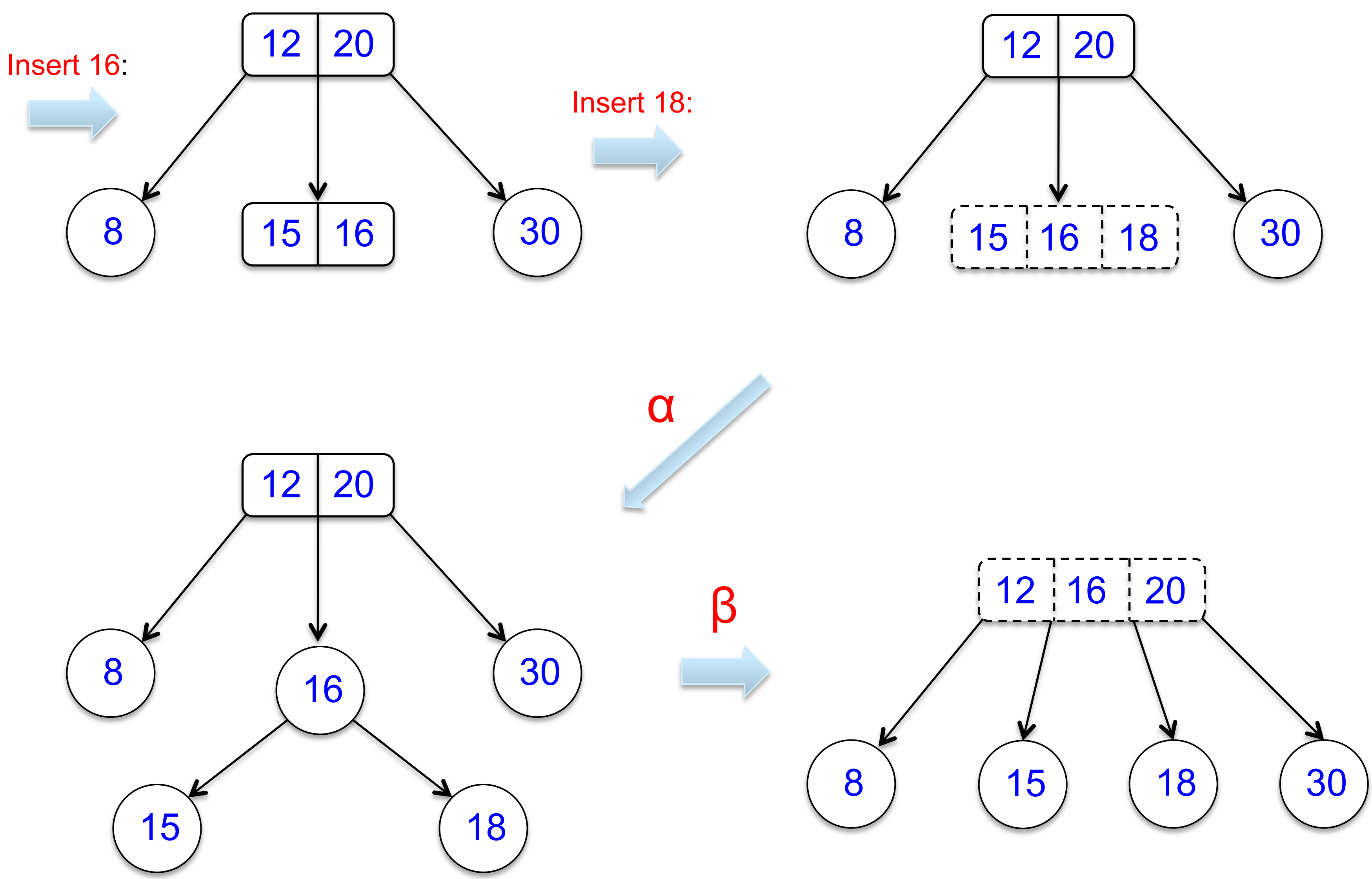
Let's continue with our example….



Insert a 16 into the tree:

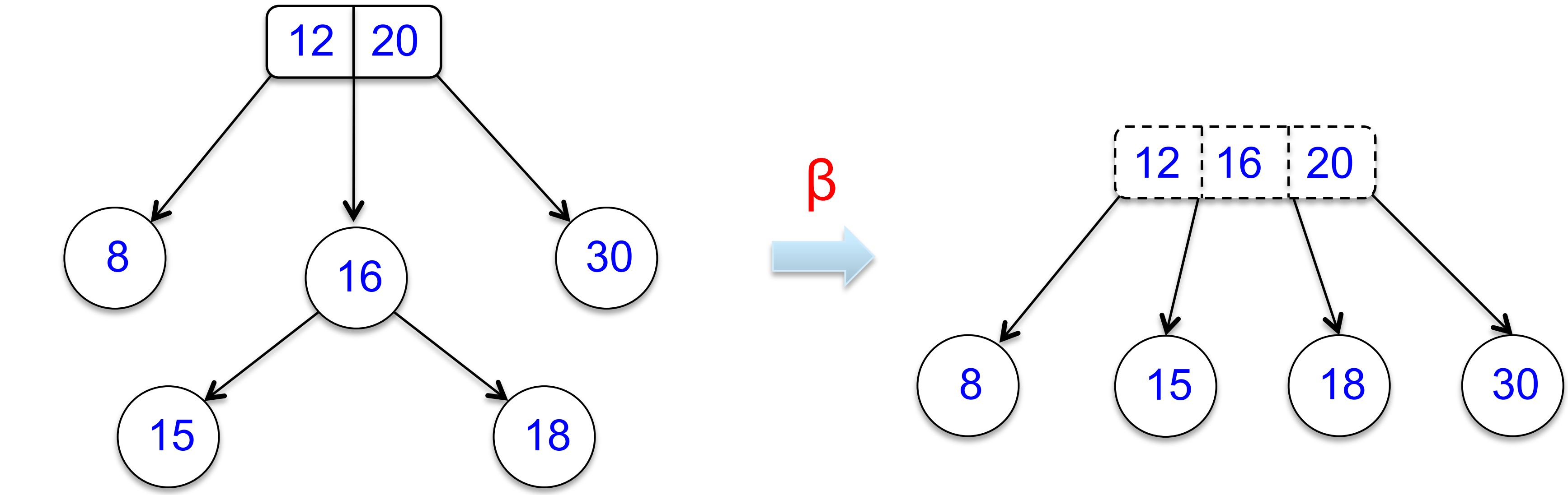


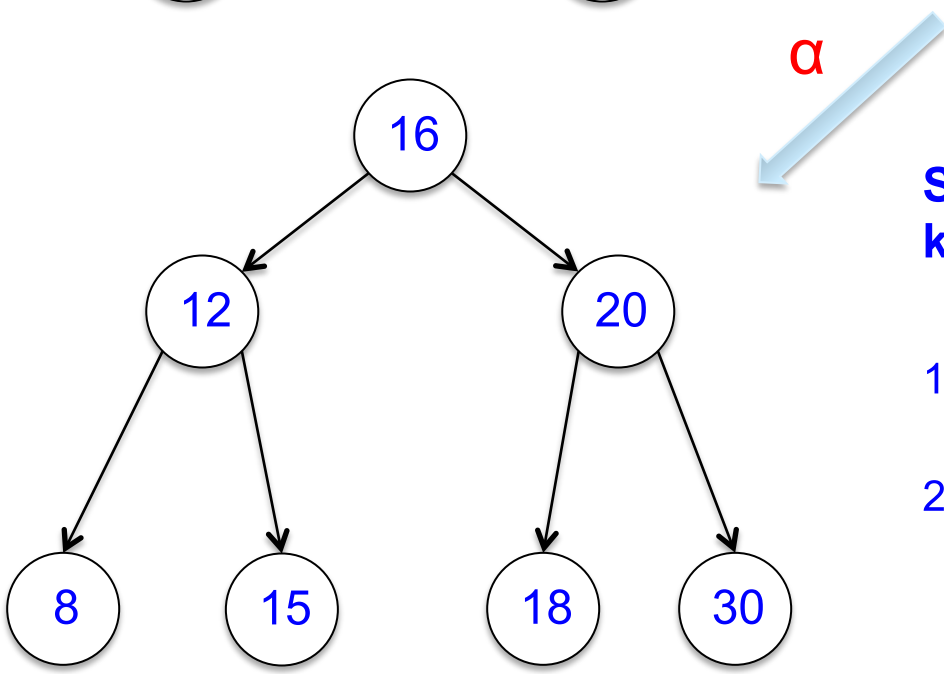
2-3 Trees



Insert 16:
12
20
8
15
16
30
Insert 18:
12
20
8
15
16
18
30
α
12
20
8
16
30
15
18
β
12
16
20
8
15
18
30

2-3 Trees





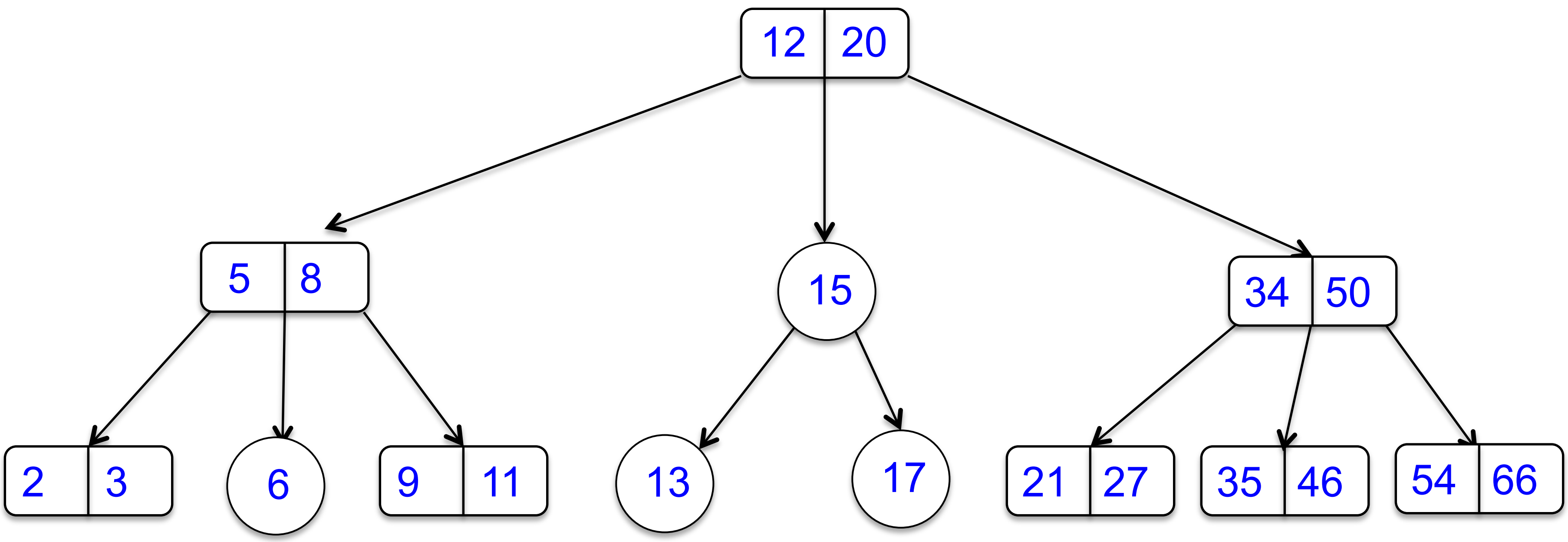
Summary of rules for inserting a new key into a 2-3 tree:

1. Insert new key into appropriate leaf node, potentially creating an error node;
2. If there is an error node, apply α- and β-transformations moving up the path to the root, until a balanced tree with no error nodes is obtained.

Worst-Case Time Complexity of 2-3 Trees (counting the number of comparisons): Member(….)

Consider the following tree:

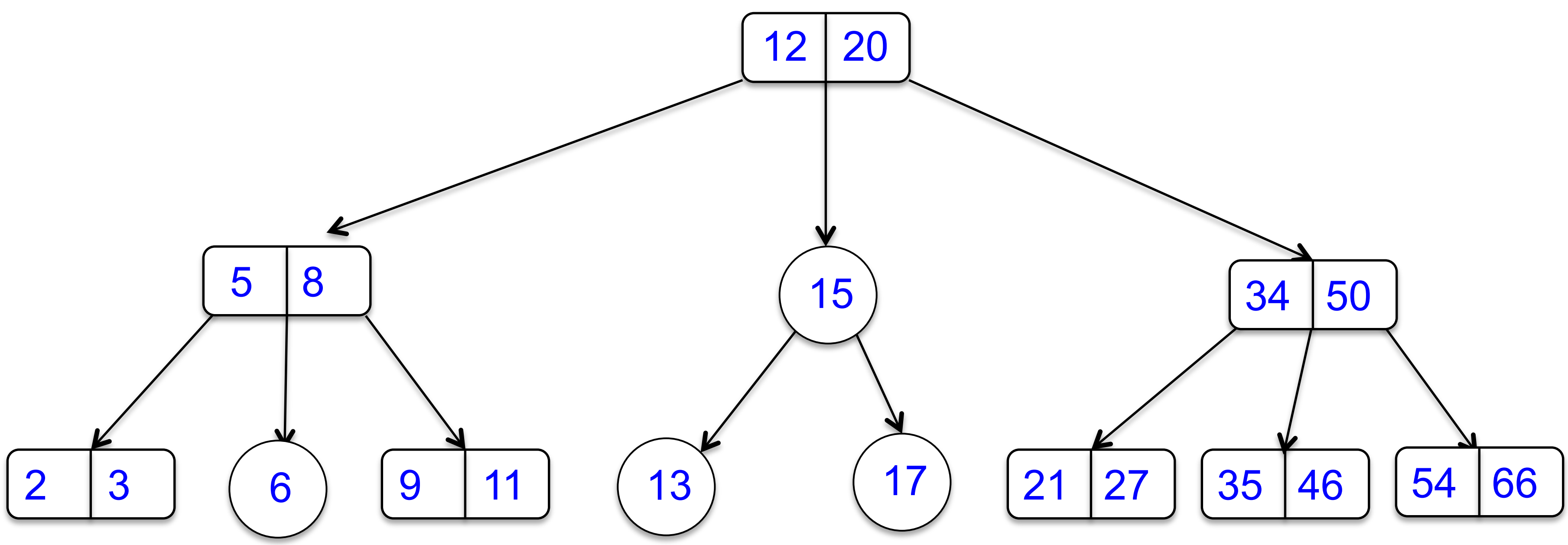
- What is the cost (# of comparisons) for finding 2?
- How about 27?
- Which keys represent the worst case for this tree?



Worst-Case Time Complexity of 2-3 Trees (counting the number of comparisons): Member(….)

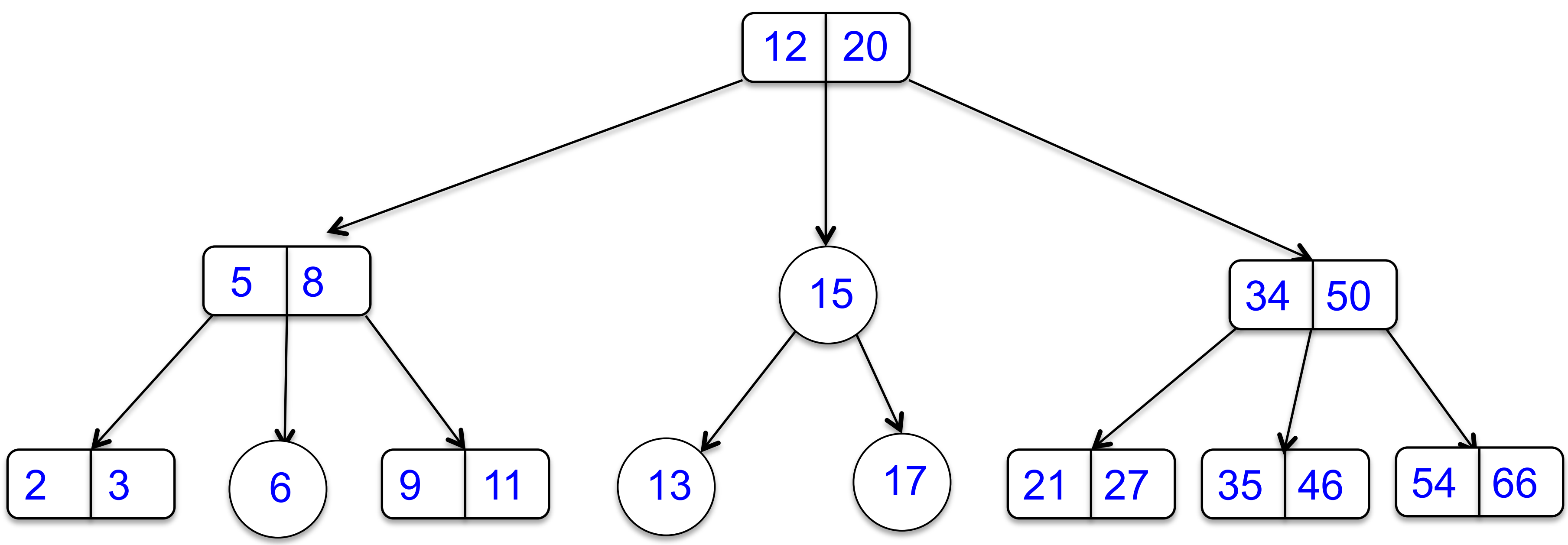
Consider the following tree:

- What is the cost (# of comparisons) for finding 2? 3
- How about 27? 5
- Which keys represent the worst case for this tree? 46 or 66, with 6 comparisons



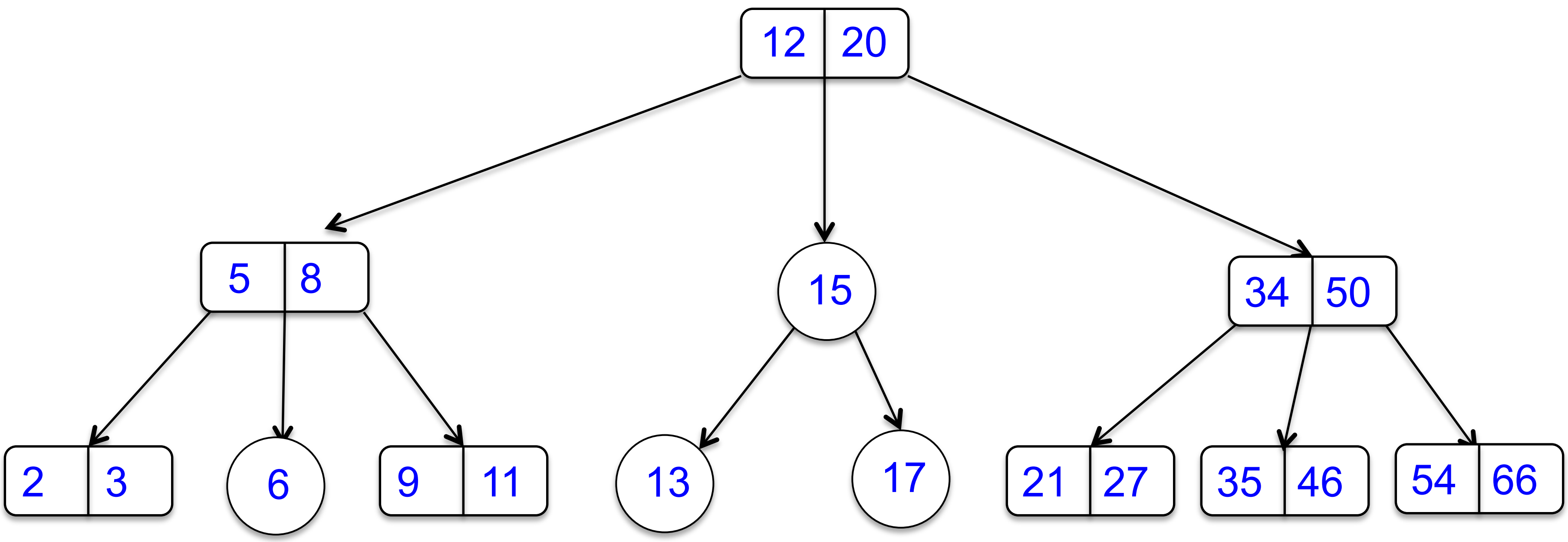
Worst-Case Time Complexity of 2-3 Trees (counting the number of comparisons): Member(….)

The worst-case for member(…) is to go all the way to a leaf node, and do 2 comparisons at each node; in a balanced tree with N keys, the height is Θ(Log N), i.e., C * Log N + …. for some constant C, but if we have to do 2 comparisons at each node, this becomes 2 * C * Log N + …. which is still Θ(Log N) comparisons.



Worst-Case Time Complexity of 2-3 Trees (counting the number of comparisons): Insert(….)

For insert(…), the worst thing that can happen is that you insert the new key at the bottom of the tree, and it causes α- and β-transformations all the way back up the tree. Each transformation takes a constant C amount of work, so the cost is Θ(Log N) to find the location (as in member(…)), and C * Θ(Log N) transform the tree back up to the root. (1 + C) * Θ(Log N) is still Θ(Log N).



2-3 Trees

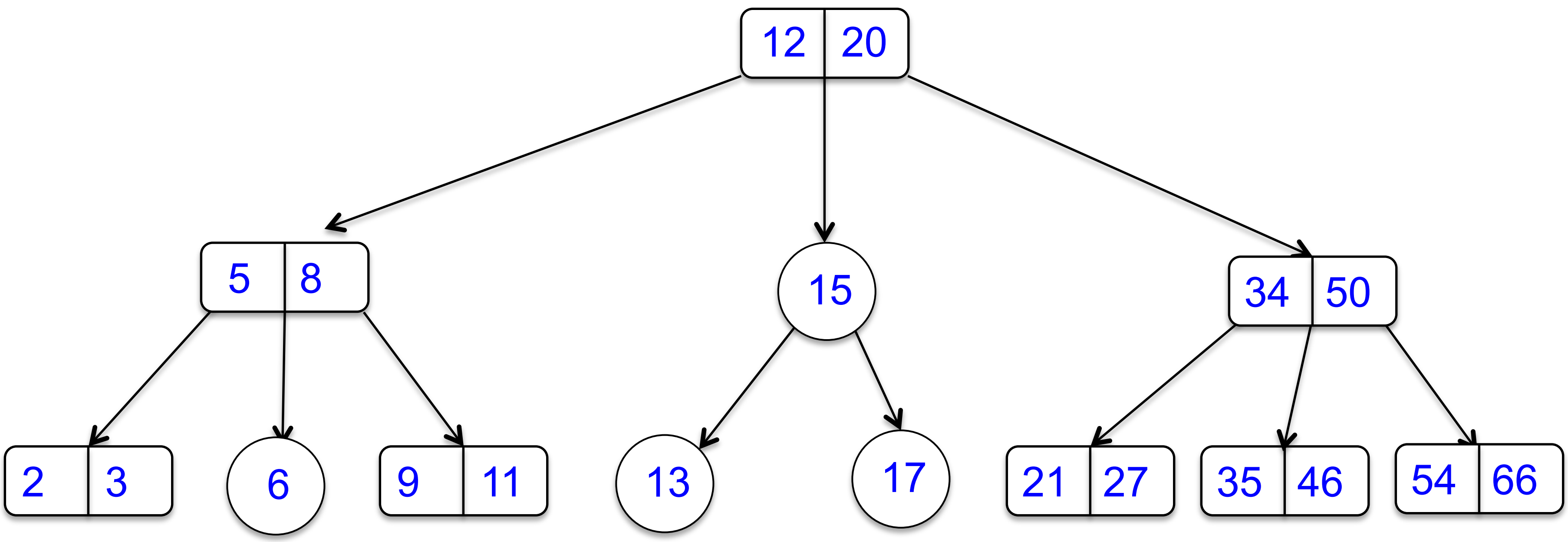
Computer Science

Worst-Case Time Complexity of 2-3 Trees (counting the number of comparisons):

Member(….): Θ(Log N)

Delete(….): Θ(Log N) (not described)

Insert(….): Θ(Log N)



2-3 Trees

Computer Science

Code Complexity: 2-3 Trees are generally encoded as normal BSTs with two different colored links ("Red-Black Trees"), and the code for insert is not as complicated as you would imagine:

```
 private static Node insert(int key, Node t) {
    if (t == null)
      return new Node(key);
    else if (key < t.key) {
      t.left = insert(key, t.left);
      return applyTransformations(t);
    } else if (key > t.key) {
      t.right = insert(key, t.right);
      return applyTransformations(t);
    } else
      return t;
  }

private static Node leanRight( Node t ) {
    Node newRoot = t.left;
    t.left = newRoot.right;
    newRoot.right = t;
    newRoot.red = t.red;
    t.red = true;
    return newRoot;
```

```
private static Node rotateLeft( Node t ) {
   Node newRoot = t.right;
   t.right = t.right.left;
   newRoot.left = t;
   newRoot.red = true;
   newRoot.left.red = false;
   newRoot.right.red = false;
   return newRoot;
 }

 private static Node applyTransformations( Node t ) {
   if(t == null)
     return null;
   if(t.left != null && t.left.red)
     t = leanRight( t );
   if(   t.right != null && t.right.red
      && t.right.right != null && t.right.right.red)
      t = rotateLeft( t );
   return t;
 }
```