Deletion in BSTs

Deletion is somewhat more complicated than insertion or lookup. We will warm up by considering a simple case first: How do we delete the minimal element in a BST?

// reconstruct the tree r without its minimal element

public static Node deleteMin(Node r) {
    if(r.left == null) {
        return r.right;
    } else {
        r.left = deleteMin(r.left);
        return r;
    }
}
Deletion in BSTs

Note for later: If we want to keep track of the minimal node (say, to remove it and use it later), we could write a simple helper function to look up the minimal element:

```java
public static Node findMin(Node r) {
    if (r.left == null) // this is the minimal node
        return r;
    else
        return findMin(r.left);
}
```

```java
public static Node deleteMin(Node r) {
    if (r.left == null)
        return r.right;
    else {
        r.left = deleteMin(r.left);
        return r;
    }
}
```

// Remove the node containing the minimal element and store it as p
Node p = findMin(root);
root = deleteMin(root);

Ok, we know how to delete the minimal element; how to delete an arbitrary element? As usual, the place to start with by enumerating all the cases, starting with null:

```java
public static Node delete(int n, Node t) {
    if (t == null) // Case 1: tree is null
        return t;
    // Case 2: 0 <= n < t.data
    else if (t.data > n) 
        t.left = delete(n, t.left);
    // Case 3: t.data < n
    else { 
        // delete t
        Node p = findMin(t.right);
        t.data = p.data;
        t.right = deleteMin(t.right);
    }
    return t;
}
```
Deletion in BSTs

```java
public static Node delete(int n, Node t) {
    if (t == null)                         // Case 1: tree is null
        return t;
    else if (n < t.key) {                  // Case 2: key n is in left subtree
        t.left = delete(n, t.left);
        return t;
    } else if (n > t.key) {                // Case 3: key n is in right subtree
        t.right = delete(n, t.right);
        return t;
    } else                          // Case 4: found key n at root;
```
Deletion in BSTs

```java
public static Node delete(int n, Node t) {
    if (t == null)                         // Case 1: tree is null
        return t;
    else if (n < t.key) {                  // Case 2: key n is in left subtree
        t.left = delete(n, t.left);
        return t;
    } else if (n > t.key) {                // Case 3: key n is in right subtree
        t.right = delete(n, t.right);
        return t;
    } else if (t.left == null)         // Case 4a: no left child, so reroute around
        return t.right;
    else if (t.right == null)        // Case 4b: no right child, ditto
        return t.left;
    else {                          // Case 4c: both children exist, so replace
        Node min = findMin(t.right);  // Case 4c: found key n at root;
        t.right = deleteMin(t.right); // root by minimal element in right subtree
        min.left = t.left;            // Find minimal node
        min.right = t.right;         // Reconstruct the right subtree without it
        t.right = min;
        return min;
    }
}
```
A Tree Traversal is an algorithm for visiting each node of a Binary Tree in some order; all algorithms which perform some kind of operation on the tree as a whole usually follow one of these traversals. We will later generalize these to traversals of arbitrary Graphs, and many problems in computer science can be phrased as traversal of some kind of graph.

Let us begin by considering how you might "explore" a tree by walking around the links (for the moment considering them as two-way) to "visit" each node at least once; one way to do this is to use a version of the famous "right-hand rule" for solving a simple maze: just start at the root, keep your right hand on the wall, and keep walking.

Instead, we will keep our left hand on the outside "wall" of the tree, and keep walking……
Instead, we will keep our left-hand on the outside “wall” of the tree, and keep walking.....

NOTE: Each node is touched exactly three times:
The nodes are touched (for the first time) in this order: F, B, A, D, C, E, G, I, H
Tree Traversals

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The nodes are touched (for the second time) in this order: A, B, C, D, E, F, G, H, I

Tree Traversals

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The nodes are touched (for the second time) in this order: A, B, C, D, E, F, G, H, I
The nodes are touched (for the third time) in this order: A, C, E, D, B, H, I, G, F
Tree Traversals

Or, we could walk around “clock-wise” instead, by using our right hand:

The nodes are visited for the first time in this order: F, G, I, H, B, D, E, C, A
The nodes are visited for the second time in this order: I, H, G, F, E, D, C, B, A
The nodes are visited for the third time in this order: H, I, G, C, D, A, B, F

How could we do this using an algorithm (which—no surprise!—will be recursive)?
As usual, we use the recursive definition of a tree as a basis for our algorithm:

A Binary Tree is either
(1) null
or:
(2) A node pointing to two Binary Trees

key
Tree Traversals

The base case is easy (do nothing!);
For the recursive case, we have to do three things:
(V) Visit the root (say, by printing out the key);
(L) Recursively traverse the left subtree; and
(R) Recursively traverse the right subtree.
It does not matter which order we do these in,
as long as we do all three.....

There are 3! = 6 possible orderings of these three lines,
giving us 6 possible recursive traversals:

Preorder - Visit root first
Inorder - Visit root second
Postorder - Visit root last

Normally, you do L before R;
If you add "Reverse" do R before L

Reverse Postorder:
R
L
V
Tree Traversals

How could we do this using an algorithm? Here is a Preorder Traversal:

```java
void traverse(Node t) {
    if (t != null) {  // Base case is implicit
        visit(t);  // V
        traverse(t.left);  // L
        traverse(t.right);  // R
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
```

Tree Traversals

Which one is this?

```java
void traverse(Node t) {
    if (t != null) {  // Base case is implicit
        traverse(t.left);  // L
        visit(t);  // V
        traverse(t.right);  // R
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
```
Tree Traversals

Which one is this? Note: L before R, Visit root in middle: Inorder

void traverse(Node t) {
    if (t != null) { // Base case is implicit
        traverse(t.left); // L
        visit(t); // V
        traverse(t.right); // R
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}

Tree Traversals

How about this?

void traverse(Node t) {
    if (t != null) { // Base case is implicit
        traverse(t.right); // R
        traverse(t.left); // L
        visit(t); // V
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
Tree Traversals

How about this? R before S (so Reverse) and Visit root last: Reverse Postorder

```java
void traverse(Node t) {
    if( t != null ) { // Base case is implicit
        traverse(t.right); // R
        traverse(t.left); // L
        visit(t); // V
    }
}

void visit(Node t) {
    System.out.print(t.key + " ");
}
```

Tree Traversals: Non-Recursive Traversals

We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

Let's try using a stack first:

```java
void traverse(Node t) {
    Stack<Node> S = new Stack<Node> ();
    S.push(t);
    while( !S.isEmpty() ) {
        Node p = S.pop();
        visit( p );
        if( p.left != null )
            S.push(p.left);
        if( p.right != null )
            S.push(p.right);
    }
}
```

What recursive traversal is this equivalent to?
We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

Let's try using a stack first:

```java
void traverse(Node t) {
    Stack<Node> S = new Stack<Node>();
    S.push(t);
    while( !S.isEmpty() ) {
        Node p = S.pop();
        visit(p);
        if( p.left != null )
            S.push(p.left);
        if( p.right != null )
            S.push(p.right);
    }
}
```

What recursive traversal is this equivalent to? Reverse Preorder

How could we get a (normal) Preorder Traversal?
Tree Traversals: Non-Recursive Traversals

We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

Let's try using a stack first:

```java
void traverse(Node t) {
    Stack<Node> S = new Stack<Node>();
    S.push(t);
    while (!S.isEmpty() ) {
        Node p = S.pop();
        visit( p );
        if( p.right!= null )
            S.push(p.right);
        if( p.left!= null )
            S.push(p.left);
    }
}
```

How could we get a (normal) Preorder Traversal?

What happens if we use a Queue instead of a stack?

```java
void traverse(Node t) {
    Queue<Node> Q = new Queue<Node>();
    Q.enqueue(t);
    while( !Q.isEmpty() ) {
        Node p = Q.dequeue();
        visit( p );
        if( p.left != null )
            Q.enqueue(p.left);
        if( p.right != null )
            Q.enqueue(p.right);
    }
}
```

Does this correspond to any of our recursive traversals?
We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

What happens if we use a Queue instead of a stack?

```java
void traverse(Node t) {
    Queue<Node> Q = new Queue<Node>();
    Q.enqueue(t);
    while( !Q.isEmpty() ) {
        Node p = Q.dequeue();
        visit(p);
        if( p.left != null )
            Q.enqueue(p.left);
        if( p.right != null )
            Q.enqueue(p.right);
    }
}
```

Does this correspond to any of our recursive traversals? NO!

This is called a Breadth-First or Level-Order Traversal.
We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

What happens if we use a **Queue** instead of a stack?

```java
void traverse(Node t) {
    Queue<Node> Q = new Queue<Node>();
    Q.enqueue(t);
    while( !Q.isEmpty() ) {
        Node p = Q.dequeue();
        visit(p);
        if( p.right != null )
            Q.enqueue(p.right);
        if( p.left != null )
            Q.enqueue(p.left);
    }
}
```

What happens if we reverse the order we enqueue the children?

---

We can traverse a tree without recursion if we use an auxiliary data structure such as a stack or queue to keep track of the path traversed.

What happens if we use a **Queue** instead of a stack?

```java
void traverse(Node t) {
    Queue<Node> Q = new Queue<Node>();
    Q.enqueue(t);
    while( !Q.isEmpty() ) {
        Node p = Q.dequeue();
        visit(p);
        if( p.right != null )
            Q.enqueue(p.right);
        if( p.left != null )
            Q.enqueue(p.left);
    }
}
```
Tree Traversals: Non-Recursive Traversals

In general, we can use any collection that supports adding and removing elements!
Suppose we have an arbitrary collection (call it Unvisited) which stores the nodes in some order:

```java
void traverse(Node t) {
    Unvisited<Node> Q = new Unvisited<Node>();
    Q.add(t);
    while (!Q.isEmpty()) {
        Node p = Q.removeNext();
        visit(p);
        if (p.left != null)
            Q.add(p.left);
        if (p.right != null)
            Q.add(p.right);
    }
}
```

We will traverse ALL the nodes, in SOME order.....
We will talk more about this when we study games...