3.1 INTRODUCTION

Probability theory is a mathematical modeling of the phenomenon of chance or randomness. If a coin is tossed in the air, it can land heads or tails, but we do not know which of these will occur in a single toss. However, suppose we repeat this experiment of tossing a coin; let \( s \) be the number of successes, that is, that a head appears, and let \( n \) be the number of tosses. Then it has been empirically observed that the ratio \( f = s/n \), called the relative frequency of the outcome, becomes stable in the long run, that is, the ratio \( f = s/n \) approaches a limit. If the coin is perfectly balanced, then we expect that the coin will land heads approximately 50 percent of the time or, in other words, the relative frequency will approach \( 1/2 \). Alternately, assuming the coin is perfectly balanced, we can arrive at the value \( 1/2 \) deductively. That is, one side of the coin is as likely to occur as the other; hence the chances of getting a head is one in two which means the probability of getting a head is \( 1/2 \). Although the specific outcome on any one toss is unknown, the behavior over the long run is determined. This stable long-run behavior of random phenomena forms the basis of probability theory.

Consider another experiment, the tossing of a six-sided die (Fig. 3-1) and observing the number of dots, or pips, that appear on the top face. Suppose the experiment is repeated \( n \) times and let \( s \) be the number of times 4 dots appear on top. Again, as \( n \) increases, the relative frequency \( f = s/n \) of the outcome 4 becomes more stable. Assuming the die is perfectly balanced, we would expect that the
stable or long-run value of this ratio is 1/6, and we say the probability of getting a 4 is 1/6. Alternately, we can arrive at the value 1/6 deductively. That is, with a perfectly balanced die, any one side of the die is as likely as any other to occur on top. Thus, the chances of getting a 4 is one in six or, in other words, the probability of getting a 4 is 1/6. Again, although the specific outcome on any one toss is unknown, the behavior over the long run is determined.

The historical development of probability theory is similar to the above discussion. That is, letting $E$ denote the outcome of an experiment, called an event, there were two ways to obtain the probability $p$ of $E$:

(a) **Classical (A Priori) Definition:** Suppose an event $E$ can occur in $s$ ways out of a total of $n$ equally likely possible ways. Then $p = s/n$.

(b) **Frequency (A Posteriori) Definition:** Suppose after $n$ repetitions, where $n$ is very large, an event $E$ occurs $s$ times. Then $p = s/n$.

Both of the above definitions have serious flaws. The classical definition is essentially circular since the idea of "equally likely" is the same as that of "with equal probability" which has not been defined. The frequency definition is not well defined since "very large" has not been defined.

The modern treatment of probability theory is axiomatic using set theory. Specifically, a mathematical model of an experiment is obtained by arbitrarily assigning probabilities to all the events, except that the assignments must satisfy certain axioms listed below. Naturally, the reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequencies. This then gives rise to problems of testing and reliability, which form the subject matter of statistics.

### 3.2 SAMPLE SPACE AND EVENTS

The set $S$ of all possible outcomes of some experiment is called the **sample space**. A particular outcome, that is, an element of $S$, is called a **sample point**. An **event** $A$ is a set of outcomes or, in other words, a subset of the sample space $S$. The event $\{a\}$ consisting of a single point $a \in S$ is called an **elementary event**. The empty set $\emptyset$ and $S$ are subsets of $S$ and hence they are events; $\emptyset$ is sometimes called the **impossible** or **null** event, and $S$ is sometimes called the **certain** or **sure** event.

Events can be combined to form new events using the various set operations:

(i) $A \cup B$ is the event that occurs iff $A$ occurs or $B$ occurs (or both).

(ii) $A \cap B$ is the event that occurs iff $A$ occurs and $B$ occurs.

(iii) $A^c$, the complement of $A$, is the event that occurs iff $A$ does not occur.

(Here "iff" is an abbreviation of "if and only if").

Events $A$ and $B$ are called **mutually exclusive** if they are disjoint, that is, if $A \cap B = \emptyset$. In other words, $A$ and $B$ are mutually exclusive if they cannot occur simultaneously. Three or more events are **mutually exclusive** if every two of them are mutually exclusive.

**EXAMPLE 3.1**

(a) **Experiment:** Toss a die and observe the number (of dots) that appears on top face.

The sample space $S$ consists of the six possible numbers, that is,

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let $A$ be the event that an even number occurs, $B$ that an odd number occurs, and $C$ that a number greater than 3 occurs, that is, let

$$A = \{2, 4, 6\}, \quad B = \{1, 3, 5\}, \quad C = \{4, 5, 6\}$$
Then:

\[ A \cup C = \{2, 4, 5, 6\} = \text{the event that an even number or a number exceeding 3 occurs} \]
\[ A \cap C = \{4, 6\} = \text{the event that an even number and a number exceeding 3 occurs} \]
\[ C' = \{1, 2, 3\} = \text{the event that a number exceeding 3 does not occur}. \]

Note that \( A \) and \( B \) are mutually exclusive, that is, \( A \cap B = \emptyset \). In other words, an even number and an odd number cannot occur simultaneously.

(b) **Experiment:** Toss a coin three times and observe the sequence of heads (\( H \)) and tails (\( T \)) that appears.

The sample space \( S \) consists of the following eight elements:

\[ S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \]

Let \( A \) be the event that two or more heads appear consecutively, and \( B \) that all the tosses are the same, that is,

\[ A = \{HHH, HHT, THH\} \quad \text{and} \quad B = \{HHH, TTT\} \]

Then \( A \cap B = \{HHH\} \) is the elementary event in which only heads appear. The event that five heads appear is the empty set \( \emptyset \).

(c) **Experiment:** Toss a coin until a head appears, and then count the number of times the coin is tossed.

The sample space of this experiment is \( S = \{1, 2, 3, \ldots, \infty\} \). Here \( \infty \) refers to the case when a head never appears, and so the coin is tossed an infinite number of times. Since every positive integer is an element of \( S \), the sample space is infinite. In fact, this is an example of a sample space which is *countably infinite*.

(d) **Experiment:** Let a pencil drop, head first, into a rectangular box and note the point at the bottom of the box that the pencil first touches. Here \( S \) consists of all the points on the bottom of the box. Let the rectangular area in Fig. 3-2 represent these points. Let \( A \) and \( B \) be the events that the pencil drops into the corresponding areas illustrated in Fig. 3-2. Then \( A \cap B \) is the event that the pencil drops in the shaded region in Fig. 3-2.

![Fig. 3-2](image)

**Remark:** The sample space \( S \) in Example 3.1(d) is an example of a continuous sample space. (A sample space \( S \) is *continuous* if it is an interval or a product of intervals.) In such a case, only special subsets (called *measurable* sets) will be events. On the other hand, if the sample space \( S \) is *discrete*, that is, if \( S \) is finite or countably infinite, then every subset of \( S \) is an event.

**EXAMPLE 3.2** Toss of a pair of dice A pair of dice is tossed and the two numbers appearing on the top faces are recorded. There are six possible numbers, 1, 2, \ldots, 6, on each die. Thus, \( S \) consists of the pairs of numbers from 1 to 6, and hence \( n(S) = 6 \cdot 6 = 36 \). Figure 3-3 shows these 36 pairs of numbers arranged in an array where the rows are labeled by the first die and the columns by the second die. Let \( A \) be the event that the sum of the two numbers is 6, and let \( B \) be the event that the largest of the two numbers is 4. That is, let

\[ A = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\} \]
\[ B = \{(1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1)\} \]
These events are pictured in Fig. 3-3. Then the event "A and B" consists of those pairs of integers whose sum is 6 and whose largest number is 4 or, in other words, the intersection of A and B. Thus

\[ A \cap B = \{(2, 4), (4, 2)\} \]

Similarly, "A or B", the sum is 6 or the largest is 4, is the union \( A \cup B \), and "not A", the sum is not 6, is the complement \( A' \).

**EXAMPLE 3.3** Deck of cards A card is drawn from an ordinary deck of 52 cards which is pictured in Fig. 3-4(a). The sample space \( S \) consists of the four suits, clubs (C), diamonds (D), hearts (H), and spades (S),
where each suit contains 13 cards which are numbered 2 to 10, and jack (J), queen (Q), king (K), and ace (A). The hearts (H) and diamonds (D) are red cards, and the spades (S) and clubs (C) are black cards. Figure 3-4(b) pictures 52 points which represent the deck $S$ of cards in the obvious way. Let $E$ be the event of a picture card, that is, a jack (J), queen (Q), or king (K), and let $F$ be the event of a heart. Then

$$E \cap F = \{JH, QH, KH\}$$

is the event of a heart and a picture card, as shaded in Fig. 3-4(b).

### 3.3 AXIOMS OF PROBABILITY

Let $S$ be a sample space, let $\mathcal{E}$ be the class of all events, and let $P$ be a real-valued function defined on $\mathcal{E}$. Then $P$ is called a probability function, and $P(A)$ is called the probability of the event $A$, when the following axioms hold:

- **[P$_1$]** For any event $A$, we have $P(A) \geq 0$.
- **[P$_2$]** For the certain event $S$, we have $P(S) = 1$.
- **[P$_3$]** For any two disjoint events $A$ and $B$, we have

  $$P(A \cup B) = P(A) + P(B)$$

- **[P$_4$]** For any infinite sequence of mutually disjoint events $A_1, A_2, A_3, \ldots$, we have

  $$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots$$

Furthermore, when $P$ does satisfy the above axioms, the sample space $S$ will be called a probability space.

The first axiom states that the probability of any event is nonnegative, and the second axiom states that the certain or sure event $S$ has probability 1. The next remarks concern the two axioms [P$_3$] and [P$_4$]. The axiom [P$_3$] formalizes the natural assumption that if $A$ and $B$ are two disjoint events, then the probability of either of them occurring is the sum of their individual probabilities. Using mathematical induction, we can then extend this additive property for two sets to any finite number of disjoint events, that is, for any mutually disjoint sets $A_1, A_2, \ldots, A_n$, we have

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \quad (\ast)$$

We emphasize that [P$_4$] does not follow from [P$_3$], even though ($\ast$) is true for every positive integer $n$. However, if the sample space $S$ is finite, then only [P$_3$] is needed, that is, [P$_4$] is superfluous.

### Theorems on Probability Spaces

The following theorems follow directly from our axioms, and will be proved here. We use $\square$ to indicate the end of a proof.

**Theorem 3.1:** The impossible event $\varnothing$, in other words, the empty set $\varnothing$ has probability zero, that is, $P(\varnothing) = 0$.

**Proof:** For any event $A$, we have $A \cup \varnothing = A$ where $A$ and $\varnothing$ are disjoint. By [P$_3$],

$$P(A) = P(A \cup \varnothing) = P(A) + P(\varnothing)$$

Adding $-P(A)$ to both sides gives $P(\varnothing) = 0$. $\square$

The next theorem, called the complement rule, formalizes our intuition that if we hit a target, say, $p = 1/3$ of the times, then we miss the target $q = 1 - p = 2/3$ of the times. [Recall that $A^c$ denotes the complement of the set $A$.]
Theorem 3.2 (Complement Rule): For any event $A$, we have

$$P(A^c) = 1 - P(A)$$

Proof: $S = A \cup A^c$ where $A$ and $A^c$ are disjoint. By $[P_2]$, $P(S) = 1$. Thus, by $[P_2]$,

$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

Adding $-P(A)$ to both sides gives us $P(A^c) = 1 - P(A)$. \qed

The next theorem tells us that the probability of any event must lie between 0 and 1. That is,

Theorem 3.3: For any event $A$, we have $0 \leq P(A) \leq 1$.

Proof: By $[P_1]$, $P(A) \geq 0$. Hence we need only show that $P(A) \leq 1$. Since $S = A \cup A^c$ where $A$ and $A^c$ are disjoint, we get

$$1 = P(S) = P(A \cup A^c) = P(A) + P(A^c)$$

Adding $-P(A^c)$ to both sides gives us $P(A) = 1 - P(A^c)$. Since $P(A^c) \geq 0$, we get $P(A) \leq 1$, as required. \qed

The following theorem applies to the case that one event is a subset of another event.

Theorem 3.4: If $A \subseteq B$, then $P(A) \leq P(B)$.

Proof: If $A \subseteq B$, then, as indicated by Fig. 3-5(a), $B = A \cup (B \setminus A)$ where $A$ and $B \setminus A$ are disjoint. Hence

$$P(B) = P(A) + P(B \setminus A)$$

By $[P_1]$, we have $P(B \setminus A) \geq 0$; hence $P(A) \leq P(B)$. \qed

![Fig. 3-5](image)

(a) $B$ is shaded. (b) $A$ is shaded. (c) $A \cup B$ is shaded.

The following theorem concerns two arbitrary events.

Theorem 3.5: For any two events $A$ and $B$, we have

$$P(A \setminus B) = P(A) - P(A \cap B)$$

Proof: As indicated by Fig. 3-5(b), $A = (A \setminus B) \cup (A \cap B)$ where $A \setminus B$ and $A \cap B$ are disjoint. Accordingly, by $[P_3]$,

$$P(A) = P(A \setminus B) + P(A \cap B)$$

from which our result follows. \qed

The next theorem, called the general addition rule, or simply addition rule, is similar to the inclusion-exclusion principle for sets.
Theorem (Addition Rule) 3.6: For any two events \( A \) and \( B \),
\[
P(A \cup B) = P(A) + P(B) - P(A \cap B)
\]

Proof: As indicated by Fig. 3-5(c), \( A \cup B = (A \setminus B) \cup B \) where \( A \setminus B \) and \( B \) are disjoint sets. Thus, using Theorem 3.5,
\[
P(A \cup B) = P(A \setminus B) + P(B) = P(A) - P(A \cap B) + P(B)
\]
\[
= P(A) + P(B) - P(A \cap B)
\]
which is our result.

Applying the above theorem twice (Problem 3.34), we obtain:

Corollary 3.7: For any events, \( A, B, C \), we have
\[
P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\]

Clearly, like the analogous inclusion-exclusion principle for sets, the addition rule can be extended by induction to any finite number of sets.

3.4 Finite Probability Spaces

Consider a finite sample space \( S \) where we assume, unless otherwise stated, that the class \( C \) of all events consists of all subsets of \( S \). As noted above, \( S \) becomes a probability space by assigning probabilities to the events in \( C \) so they satisfy the probability axioms. This section shows how this is usually done when the sample space \( S \) is finite. The next section discusses infinite sample spaces.

Finite Equiprobable Spaces

Suppose \( S \) is a finite sample space, and suppose the physical characteristics of the experiment suggest that the various outcomes of the experiment be assigned equal probabilities. Such a probability space \( S \), where each point is assigned the same probability, is called a finite equiprobable space. Specifically, if \( S \) has \( n \) elements, then each point in \( S \) is assigned the probability \( 1/n \) and each event \( A \) containing \( r \) points is assigned the probability \( r/n \). In other words,
\[
P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)}
\]
or
\[
P(A) = \frac{\text{number of ways that the event } A \text{ can occur}}{\text{number of ways that the sample space } S \text{ can occur}}
\]

We emphasize that the above formula for \( P(A) \) can only be used with respect to an equiprobable space, and cannot be used in general.

We state the above result formally.

Theorem 3.8: Let \( S \) be a finite sample space and, for any subset \( A \) of \( S \), let \( P(A) = n(A)/n(S) \). Then \( P \) satisfies axioms \([P_1]\), \([P_2]\), and \([P_3]\).

The expression "at random" will be used only with respect to an equiprobable space; formally, the statement "choose a point at random from a set \( S \)" shall mean that \( S \) is an equiprobable space where each point in \( S \) has the same probability.
EXAMPLE 3.4 A card is selected at random from an ordinary deck of 52 playing cards. (See Fig. 3-4.) Consider the following events [where a face card is a jack (J), queen (Q), or king (K)]:

\[ A = \{\text{heart}\} \quad \text{and} \quad B = \{\text{face card}\} \]

(a) Find \( P(A) \), \( P(B) \), and \( P(A \cap B) \). (b) Find \( P(A \cup B) \).

(a) Since we have an equiprobable space,

\[
P(A) = \frac{\text{number of hearts}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4}, \quad P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}.
\]

\[
P(A \cap B) = \frac{\text{number of heart face cards}}{\text{number of cards}} = \frac{3}{52}.
\]

(b) Since we want \( P(A \cup B) \), the probability that the card is a heart or a face card, we can count the number of such cards and use Theorem 3.8. Alternately, we can use (a) and the Addition Rule Theorem 3.6 to obtain

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{4} + \frac{3}{13} - \frac{3}{52} = \frac{11}{26}.
\]

EXAMPLE 3.5 Suppose a student is selected at random from 80 students where 30 are taking mathematics, 20 are taking chemistry, and 10 are taking mathematics and chemistry. Find the probability \( p \) that the student is taking mathematics (\( M \)) or chemistry (\( C \)).

Since the space is equiprobable, we have:

\[
P(M) = \frac{30}{80} = \frac{3}{8}, \quad P(C) = \frac{20}{80} = \frac{1}{4}, \quad P(M \text{ and } C) = P(M \cap C) = \frac{10}{80} = \frac{1}{8}.
\]

Thus, by the Addition Rule (Theorem 3.6),

\[
p = P(M \text{ or } C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{8} + \frac{1}{4} - \frac{1}{8} = \frac{1}{2}.
\]

Finite Probability Spaces

Let \( S \) be a finite sample space, say \( S = \{a_1, a_2, \ldots, a_n\} \). A finite probability space, or finite probability model, is obtained by assigning to each point \( a_i \) in \( S \) a real number \( p_i \), called the probability of \( a_i \), satisfying the following properties:

(i) Each \( p_i \) is nonnegative, that is, \( p_i \geq 0 \).

(ii) The sum of the \( p_i \) is 1, that is,

\[
\sum p_i = p_1 + p_2 + \cdots + p_n = 1
\]

The probability \( P(A) \) of an event \( A \) is defined as the sum of the probabilities of the points in \( A \), that is,

\[
P(A) = \sum_{a_i \in A} P(a_i) = \sum_{a_i \in A} p_i.
\]

For notational convenience, we write \( P(a_i) \) instead of \( P(\{a_i\}) \).

Sometimes the points in a finite sample space \( S \) and their assigned probabilities are given in the form of a table as follows:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>\cdots</th>
<th>( a_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>( p_1 )</td>
<td>( p_2 )</td>
<td>\cdots</td>
<td>( p_n )</td>
</tr>
</tbody>
</table>

Such a table is called a probability distribution.
The fact that \( P(A) \), the sum of the probabilities of the points in \( A \), does define a probability space is stated formally below (and proved in Problem 3.32).

**Theorem 3.9:** The above function \( P(A) \) satisfies the axioms

\([P_1], [P_2], \text{and } [P_3].\)

**EXAMPLE 3.6 **Experiment Let three coins be tossed and the number of heads observed. [Compare with Example 3.1(b).] Then the sample space is \( S = \{0, 1, 2, 3\} \). The following assignments on the elements of \( S \) define a probability space:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>1/8</td>
<td>3/8</td>
<td>3/8</td>
<td>1/8</td>
</tr>
</tbody>
</table>

That is, each probability is nonnegative, and the sum of the probabilities is 1. Let \( A \) be the event that at least one head appears, and let \( B \) be the event that all heads or all tails appear, that is, let

\[ A = \{1, 2, 3\} \quad \text{and} \quad B = \{0, 3\} \]

Then, by definition,

\[ P(A) = P(1) + P(2) + P(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8} \]

and

\[ P(B) = P(0) + P(3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \]

**EXAMPLE 3.7** Three horses \( A, B, \) and \( C \) are in a race; \( A \) is twice as likely to win as \( B \), and \( B \) is twice as likely to win as \( C \).

(a) Find their respective probabilities of winning, that is, find \( P(A), P(B), P(C) \).

(b) Find the probability that \( B \) or \( C \) wins.

(a) Let \( P(C) = p \). Since \( B \) is twice as likely to win as \( C \), \( P(B) = 2p \); and since \( A \) is twice as likely to win as \( B \), \( P(A) = 2P(B) = 2(2p) = 4p \). Now the sum of the probabilities must be 1; hence

\[ p + 2p + 4p = 1 \quad \text{or} \quad 7p = 1 \quad \text{or} \quad p = \frac{1}{7} \]

Accordingly, \( P(A) = 4p = \frac{4}{7} \), \( P(B) = 2p = \frac{2}{7} \), \( P(C) = p = \frac{1}{7} \).

(b) Note \( \{B, C\} \) is the event that \( B \) or \( C \) wins, so we want \( P(\{B, C\}) \). By definition, we simply add up the probabilities of the points in \( \{B, C\} \). Thus

\[ P(\{B, C\}) = P(B) + P(C) = \frac{2}{7} + \frac{1}{7} = \frac{3}{7} \]

### 3.5 INFINITE SAMPLE SPACES

This section considers infinite sample spaces \( S \). There are two cases, the case where \( S \) is countably infinite and the case where \( S \) is uncountable. We note that a finite or a countably infinite probability space \( S \) is said to be *discrete*, whereas an uncountable space \( S \) is said to be *nondiscrete*. Moreover, an uncountable space \( S \) which consists of a continuum of points, such as an interval or product of intervals, is said to be *continuous*. 
Countably Infinite Sample Spaces

Suppose \( \mathcal{S} \) is a countably infinite sample space; say
\[
\mathcal{S} = \{a_1, a_2, a_3, \ldots \}
\]
Then, as in the finite case, we obtain a probability space by assigning each \( a_i \in \mathcal{S} \) a real number \( p_i \), called its probability, such that:

(i) Each \( p_i \) is nonnegative, that is, \( p_i \geq 0 \).

(ii) The sum of the \( p_i \) is equal to 1, that is,
\[
p_1 + p_2 + p_3 + \cdots = \sum_{i=1}^{\infty} p_i = 1
\]
The probability \( P(A) \) of an event \( A \) is then the sum of the probabilities of its points.

**EXAMPLE 3.8** Consider the sample space \( \mathcal{S} = \{1, 2, 3, \ldots, n\} \) of the experiment of tossing a coin until a head appears; here \( n \) denotes the number of times the coin is tossed. A probability space is obtained by setting
\[
p(1) = \frac{1}{2}, \quad p(2) = \frac{1}{4}, \quad p(3) = \frac{1}{8}, \ldots, \quad p(n) = \frac{1}{2^n}, \ldots, \quad p(\infty) = 0
\]
Consider the events:
\[
A = \{n \text{ is at most 3}\} = \{1, 2, 3\} \quad \text{and} \quad B = \{n \text{ is even}\} = \{2, 4, 6, \ldots\}
\]
Find \( P(A) \) and \( P(B) \).
Adding the probabilities of the points in the sets (events) yields:
\[
P(A) = P(1, 2, 3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}
\]
\[
P(B) = P(2, 4, 6, \ldots) = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots
\]
Note that \( P(B) \) is a geometric series with \( a = 1/4 \) and \( r = 1/4 \); hence
\[
P(B) = \frac{a}{1 - r} = \frac{1/4}{3/4} = \frac{1}{3}
\]

Uncountable Spaces

The only uncountable sample spaces \( \mathcal{S} \) which we will consider here are those with some finite geometrical measurement \( m(S) \), such as length, area, or volume, and where a point in \( \mathcal{S} \) is selected at random. The probability of an event \( A \), that is, that the selected point belongs to \( A \), is then the ratio of \( m(A) \) to \( m(S) \). Thus
\[
P(A) = \frac{\text{length of } A}{\text{length of } \mathcal{S}} \quad \text{or} \quad P(A) = \frac{\text{area of } A}{\text{area of } \mathcal{S}} \quad \text{or} \quad P(A) = \frac{\text{volume of } A}{\text{volume of } \mathcal{S}}
\]
Such a probability space \( \mathcal{S} \) is said to be uniform.

**EXAMPLE 3.9** A point is chosen at random inside a rectangle measuring 3 by 5 in. Find the probability \( p \) that the point is at least 1 in from the edge.
Let \( S \) denote the set of points inside the rectangle and let \( A \) denote the set of points at least 1 in from the edge. \( S \) and \( A \) are pictured in Fig. 3-6. Note that \( A \) is a rectangular area measuring 1 in by 3 in. Thus
\[
p = \frac{\text{area of } A}{\text{area of } S} = \frac{1 \cdot 3}{3 \cdot 5} = \frac{1}{5}
\]
3.6 CLASSICAL BIRTHDAY PROBLEM

The classical birthday problem concerns the probability that \( n \) people have distinct birthdays where \( n \leq 365 \). Here we ignore leap years and assume that a person's birthday can fall on any day with equal probability.

Since there are \( n \) people and 365 different days, there are \( 365^n \) ways in which the \( n \) people can have their birthdays. On the other hand, if the \( n \) persons are to have distinct birthdays, then:

(i) The first person can be born on any of the 365 days.
(ii) The second person can be born on the remaining 364 days.
(iii) The third person can be born on the remaining 363 days, and so on.

Thus there are:

\[
365 \cdot 364 \cdot 363 \cdots (365 - n + 1)
\]

ways that \( n \) persons can have distinct birthdays. Therefore

\[
P(\text{n people have distinct birthdays}) = \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}
\]

Accordingly, the probability \( p \) that two or more people have the same birthday is as follows:

\[
p = 1 - [\text{probability that no two people have the same birthday}]
= 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}
\]

The value of \( p \) where \( n \) is a multiple of 10 up to 60 follows:

<table>
<thead>
<tr>
<th>( n )</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p )</td>
<td>0.117</td>
<td>0.411</td>
<td>0.706</td>
<td>0.891</td>
<td>0.970</td>
<td>0.994</td>
</tr>
</tbody>
</table>

We note that \( p = 0.476 \) for \( n = 22 \) and that \( p = 0.507 \) for \( n = 23 \). Accordingly:

In a group of 23 people, it is more likely that at least two of them have the same birthday than that they all have distinct birthdays.

The above table also tells us that, in a group of 60 or more people, the probability that two or more of them have the same birthday exceeds 99 percent.
Solved Problems

SAMPLE SPACES AND EVENTS

3.1. Let A and B be events. Find an expression and exhibit the Venn diagram for the event:

(a) A but not B, (b) neither A nor B, (c) either A or B, but not both.

(a) Since A but not B occurs, shade the area of A outside of B, as in Fig. 3-7(a). Note that $B^c$, the complement of B, occurs, since B does not occur; hence $A$ and $B^c$ occur. In other words, the event is $A \cap B^c$.

(b) "Neither A nor B" means "not A and not B" or $A^c \cap B^c$. By DeMorgan's law, this is also the set $(A \cup B)^c$; hence shade the area outside of A and outside of B, that is, outside $A \cup B$, as in Fig. 3-7(b).

(c) Since A or B, but not both, occurs, shade the area of A and B, except where they intersect, as in Fig. 3-7(c). The event is equivalent to the occurrence of A but not B or B but not A. Thus, the event is $(A \cap B^c) \cup (B \cap A^c)$. Alternately, the event is $A \oplus B$, the symmetric difference of A and B.

Fig. 3-7

3.2. Let A, B, C be events. Find an expression and exhibit the Venn diagram for the event:

(a) A and B but not C occurs, (b) only A occurs.

(a) Since A and B but not C occurs, shade the intersection of A and B which lies outside of C, as in Fig. 3-8(a). The event consists of the elements in A, in B, and in $C^c$ (not in C), that is, the event is the intersection $A \cap B \cap C^c$.

(b) Since only A is to occur, shade the area of A which lies outside of B and C, as in Fig. 3-8(b). The event consists of the elements in A, in $B^c$ (not in B), and in $C^c$ (not in C), that is, the event is the intersection $A \cap B^c \cap C^c$.

Fig. 3-8
3.3. Let a coin and a die be tossed; and let the sample space \( S \) consist of the 12 elements:

\[ S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\} \]

Express explicitly the following events:

(a) \( A = \{\text{heads and an even number}\} \), (b) \( B = \{\text{a number less than 3}\} \),
(c) \( C = \{\text{tails and an odd number}\} \).

(a) The elements of \( A \) are those elements of \( S \) which consist of an \( H \) and an even number; hence

\[ A = \{H2, H4, H6\} \]

(b) The elements of \( B \) are those elements of \( S \) whose second component is less than 3, that is, 1 or 2; hence

\[ B = \{H1, H2, T1, T2\} \]

(c) The elements of \( C \) are those elements of \( S \) which consist of a \( T \) and an odd number; hence

\[ C = \{T1, T3, T5\} \]

3.4. Consider the events \( A, B, C \) in the preceding Problem 3.3. Express explicitly the event that:

(a) \( A \) or \( B \) occurs. (b) \( B \) and \( C \) occur. (c) Only \( B \) occurs.

Which pair of the events \( A, B, C \) are mutually exclusive?

(a) "\( A \) or \( B \)" is the set union \( A \cup B \); hence

\[ A \cup B = \{H2, H4, H6, H1, T1, T2\} \]

(b) "\( B \) and \( C \)" is the set intersection \( B \cap C \); hence

\[ B \cap C = \{T1\} \]

(c) "Only \( B \)" consists of the elements of \( B \) which are not in \( A \) and not in \( B \), that is, the set intersection \( B \cap A^c \cap C^c \); hence

\[ B \cap A^c \cap C^c = \{H1, T2\} \]

Only \( A \) and \( C \) are mutually exclusive, that is, \( A \cap C = \emptyset \).

3.5. A pair of dice is tossed and the two numbers appearing on the top are recorded. Recall that \( S \) consists of 36 pairs of numbers which are pictured in Fig. 3-3. Find the number of elements in each of the following events:

(a) \( A = \{\text{two numbers are equal}\} \) \quad (c) \( C = \{\text{5 appears on first die}\} \)
(b) \( B = \{\text{sum is 10 or more}\} \) \quad (d) \( D = \{\text{5 appears on at least one die}\} \)

Use Fig. 3-3 to help count the number of elements in each of the events:

(a) \( A = \{(1, 1), (2, 2), \ldots, (6, 6)\} \), so \( n(A) = 6 \).
(b) \( B = \{(6, 4), (5, 5), (4, 6), (6, 5), (5, 6), (6, 6)\} \), so \( n(B) = 6 \).
(c) \( C = \{(5, 1), (5, 2), \ldots, (5, 6)\} \), so \( n(C) = 6 \).
(d) There are six pairs with 5 as the first element, and six pairs with 5 as the second element. However, (5, 5) appears in both places. Hence

\[ n(D) = 6 + 6 - 1 = 11 \]

Alternately, count the pairs in Fig. 3-3 which are in \( D \) to get \( n(D) = 11 \).
**FINITE EQUIPROBABLE SPACES**

3.6. Determine the probability \( p \) of each event:

(a) An even number appears in the toss of a fair die.
(b) At least one tail appears in the toss of 3 fair coins.
(c) A white marble appears in the random drawing of 1 marble from a box containing 4 white, 3 red, and 5 blue marbles.

Each sample space \( S \) is an equiprobable space. Hence, for each event \( E \), use

\[
P(E) = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{n(E)}{n(S)}
\]

(a) The event can occur in three ways (2, 4, or 6) out of 6 equally like cases; hence \( p = \frac{3}{6} = \frac{1}{2} \).
(b) Assuming the coins are distinguished, there are 8 cases:

\[
HHH, HHT, HTH, HTT, THH, THT, TTH, TTT
\]

Only the first case is not favorable; hence \( p = \frac{7}{8} \).
(c) There are \( 4 + 3 + 5 = 12 \) marbles of which 4 are white; hence \( p = \frac{4}{12} = \frac{1}{3} \).

3.7. A single card is drawn from an ordinary deck \( S \) of 52 cards. (See Fig. 3-4.) Find the probability \( p \) that the card is a: (a) king, (b) face card (jack, queen, or king), (c) red card (heart or diamond), (d) red face card.

Here \( n(S) = 52 \).

(a) There are 4 kings; hence \( p = \frac{4}{52} = \frac{1}{13} \).
(b) There are \( 4(3) = 12 \) face cards; hence \( p = \frac{12}{52} = \frac{3}{13} \).
(c) There are 13 hearts and 13 diamonds; hence \( p = \frac{26}{52} = \frac{1}{2} \).
(d) There are 6 face cards which are red; hence \( p = \frac{6}{52} = \frac{3}{26} \).

3.8. Consider the sample space \( S \) and events \( A, B, C \) in Problem 3.3 where a coin and a die are tossed. Suppose the coin and die are fair; hence \( S \) is an equiprobable space. Find:

(a) \( P(A), P(B), P(C) \),
(b) \( P(A \cup B), P(B \cap C), P(B \cap A^c \cap C^c) \)

Since \( S \) is an equiprobable space, use \( P(E) = \frac{n(E)}{n(S)} \). Here \( n(S) = 12 \). We need only count the number of elements in each given set, and then divide by 12.

(a) By Problem 3.3, \( P(A) = \frac{3}{12}, P(B) = \frac{4}{12}, P(C) = \frac{3}{12} \).
(b) By Problem 3.4, \( P(A \cup B) = \frac{6}{12}, P(B \cap C) = \frac{1}{12}, P(B \cap A^c \cap C^c) = \frac{2}{12} \).

3.9. A box contains 15 billiard balls which are numbered from 1 to 15. A ball is drawn at random and the number recorded. Find the probability \( p \) that the number is:

(a) even, \( b \) less than 5, \( c \) even and less than 5, \( d \) even or less than 5.

(a) There are 7 numbers, 2, 4, 6, 8, 10, 12, 14, which are even; hence \( p = \frac{7}{15} \).
(b) There are 4 numbers, 1, 2, 3, 4, which are less than 5; hence \( p = \frac{4}{15} \).
(c) There are 2 numbers, 2 and 4, which are even and less than 5; hence \( p = \frac{2}{15} \).
(d) By the addition rule (Theorem 3.6),

\[
p = \frac{7}{15} + \frac{4}{15} - \frac{2}{15} = \frac{9}{15}
\]

Alternately, there are 9 numbers, 1, 2, 3, 4, 6, 8, 10, 12, 14, which are even or less than 5; hence \( p = \frac{9}{15} \).
3.10. A box contains 2 white socks and 2 blue socks. Two socks are drawn at random. Find the probability \( p \) they are a match (same color).

There are \( C(4, 2) = \binom{4}{2} = 6 \) ways to draw 2 of the socks. Only two pairs will yield a match. Thus \( p = 2/6 = 1/3 \).

3.11. Five horses are in a race. Audrey picks 2 of the horses at random and bets on them. Find the probability \( p \) that Audrey picked the winner.

There are \( C(5, 2) = \binom{5}{2} = 10 \) ways to pick 2 of the horses. Four of the pairs will contain the winner. Thus, \( p = 4/10 = 2/5 \).

3.12. A class contains 10 men and 20 women of which half the men and half the women have brown eyes. Find the probability \( p \) that a person chosen at random is a man or has brown eyes.

Let \( A = \{ \text{men} \} \), \( B = \{ \text{brown eyes} \} \). We seek \( P(A \cup B) \). First find:

\[
P(A) = \frac{10}{30} = \frac{1}{3}, \quad P(B) = \frac{15}{30} = \frac{1}{2}, \quad P(A \cap B) = \frac{5}{30} = \frac{1}{6}
\]

Thus, by the addition rule (Theorem 3.6),

\[
P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{2}{3}
\]

3.13. Six married people are standing in a room. Two people are chosen at random. Find the probability \( p \) that: (a) they are married; (b) one is male and one is female.

There are \( C(12, 2) = 66 \) ways to choose 2 people from the 12 people.

(a) There are 6 married couples; hence \( p = 6/66 = 1/11 \).

(b) There are 6 ways to choose the male and 6 ways to choose the female; hence \( p = (6 \cdot 6)/66 = 36/66 = 6/11 \).

3.14. Suppose 5 marbles are placed in 5 boxes at random. Find the probability \( p \) that exactly 1 of the boxes is empty.

There are exactly \( 5^4 \) ways to place the 5 marbles in the 5 boxes. If exactly 1 box is empty, then 1 box contains 2 marbles and each of the remaining boxes contains 1 marble. There are 5 ways to select the empty box, then 4 way to select the box containing 2 marbles, and \( C(5, 2) = 10 \) ways to select 2 marbles to go into this box. Finally, there are 3! ways to distribute the remaining 3 marbles among the remaining 3 boxes. Thus

\[
p = \frac{5 \cdot 4 \cdot 10 \cdot 3!}{5^5} = \frac{48}{125}
\]
3.15. Two cards are drawn at random from an ordinary deck of 52 cards. (See Fig. 3-4.) Find the probability $p$ that: (a) both are hearts, (b) one is a heart and one is a spade.

There are $C(52, 2) = 1326$ ways to choose 2 cards from the 52-card deck. In other words, $n(S) = 1326$.

(a) There are $C(13, 2) = 78$ ways to draw 2 hearts from the 13 hearts; hence

$$p = \frac{\text{number of ways 2 hearts can be drawn}}{\text{number of ways 2 cards can be drawn}} = \frac{78}{1326} = \frac{3}{51}$$

(b) There are 13 hearts and 13 spades, so there are $13 \cdot 13 = 169$ ways to draw a heart and a spade. Thus, $p = \frac{169}{1326} = \frac{13}{102}$.

FINITE PROBABILITY SPACES

3.16. A sample space $S$ consists of four elements, that is, $S = \{a_1, a_2, a_3, a_4\}$. Under which of the following functions $P$ does $S$ become a probability space?

(a) $P(a_1) = 0.4$, $P(a_2) = 0.3$, $P(a_3) = 0.2$, $P(a_4) = 0.3$.

(b) $P(a_1) = 0.4$, $P(a_2) = 0.2$, $P(a_3) = 0.7$, $P(a_4) = 0.1$.

(c) $P(a_1) = 0.4$, $P(a_2) = 0.2$, $P(a_3) = 0.1$, $P(a_4) = 0.3$.

(d) $P(a_1) = 0.4$, $P(a_2) = 0$, $P(a_3) = 0.5$, $P(a_4) = 0.1$.

(a) The sum of the values on the points in $S$ exceeds one; hence $P$ does not define $S$ to be a probability space.

(b) Since $P(a_2)$ is negative, $P$ does not define $S$ to be a probability space.

(c) Each value is nonnegative and their sum is one; hence $P$ does define $S$ to be a probability space.

(d) Although $P(a_2) = 0$, each value is still nonnegative and their sum does equal. Thus, $P$ does define $S$ to be a probability space.

3.17. A coin is weighted so that heads is twice as likely to appear as tails. Find $P(T)$ and $P(H)$.

Let $P(T) = p$; then $P(H) = 2p$. Now set the sum of the probabilities equal to one, that is, set $p + 2p = 1$. Then $p = 1/3$. Thus $P(H) = 1/3$ and $P(B) = 2/3$.

3.18. Suppose $A$ and $B$ are events with $P(A) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Find the probability that:

(a) $A$ does not occur.

(b) $B$ does not occur.

(c) $A$ or $B$ occurs.

(d) Neither $A$ nor $B$ occurs.

(a) By the complement rule, $P(\text{not } A) = P(A^c) = 1 - P(A) = 0.4$.

(b) By the complement rule, $P(\text{not } B) = P(B^c) = 1 - P(B) = 0.7$.

(c) By the addition rule,

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.3 - 0.2 = 0.7$$

(d) Recall [Fig. 3-7(b)] that neither $A$ nor $B$ is the complement of $A \cup B$. Therefore

$$P(\text{neither } A \text{ nor } B) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.7 = 0.3$$
3.19. A die is weighted so that the outcomes produce the following probability distribution:

<table>
<thead>
<tr>
<th>Outcome</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>0.1</td>
<td>0.3</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Consider the events:

\[ A = \{\text{even number}\}, \quad B = \{2, 3, 4, 5\}, \quad C = \{x : x < 3\}, \quad D = \{x : x > 7\} \]

Find the following probabilities:

(a) \( P(A) \)  
(b) \( P(B) \)  
(c) \( P(C) \)  
(d) \( P(D) \)

For any event \( E \), find \( P(E) \) by summing the probabilities of the elements in \( E \).

(a) \( A = \{2, 4, 6\} \), so \( P(A) = 0.3 + 0.1 + 0.2 = 0.6 \).
(b) \( P(B) = 0.3 + 0.2 + 0.1 + 0.1 = 0.7 \).
(c) \( C = \{1, 2\} \), so \( P(C) = 0.1 + 0.3 = 0.4 \).
(d) \( D = \emptyset \), the empty set. Hence \( P(D) = 0 \).

3.20. For the data in Problem 3.19, find:

(a) \( P(A \cap B) \)  
(b) \( P(A \cup C) \)  
(c) \( P(B \cap C) \)

First find the elements in the event, and then add the probabilities of the elements.

(a) \( A \cap B = \{2, 4\} \), so \( P(A \cap B) = 0.3 + 0.1 = 0.4 \).
(b) \( A \cup C = \{1, 2, 3, 4, 5\} \), so \( P(A \cup C) = 1 - 0.2 = 0.8 \).
(c) \( B \cap C = \{2\} \), so \( P(B \cap C) = 0.3 \).

3.21. Let \( A \) and \( B \) be events such that \( P(A \cup B) = 0.8 \), \( P(A) = 0.4 \), and \( P(A \cap B) = 0.3 \). Find:

(a) \( P(A^c) \)  
(b) \( P(B) \)  
(c) \( P(A \cap B^c) \)  
(d) \( P(A^c \cap B^c) \)

(a) By the complement rule, \( P(A^c) = 1 - P(A) = 1 - 0.4 = 0.6 \).
(b) By the addition rule, \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \). Substitute in this formula to obtain:

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.8 \]

or

\[ P(B) = 0.1 \]

(c) \( P(A \cap B^c) = P(A \setminus B) = P(A) - P(A \cap B) = 0.4 - 0.3 = 0.1 \).
(d) By DeMorgan's law, \( (A \cup B)^c = A^c \cap B^c \). Thus

\[ P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 1 - 0.8 = 0.2 \]

3.22. Suppose \( S = \{a_1, a_2, a_3, a_4\} \), and suppose \( P \) is a probability function defined on \( S \).

(a) Find \( P(a_1) \) if \( P(a_2) = 0.4 \), \( P(a_3) = 0.2 \), \( P(a_1) = 0.1 \).
(b) Find \( P(a_1) \) and \( P(a_2) \) if \( P(a_3) = P(a_4) = 0.2 \) and \( P(a_1) = 3P(a_2) \).

(a) Let \( P(a_1) = p \). For \( P \) to be a probability function, the sum of the probabilities on the sample points must equal one. Thus, we have

\[ p + 0.4 + 0.2 + 0.1 = 1 \quad \text{or} \quad p = 0.3 \]

(b) Let \( P(a_2) = p \) so \( P(a_1) = 3p \). Thus

\[ 3p + p + 0.2 + 0.2 = 1 \quad \text{or} \quad p = 0.15 \]

Hence \( P(a_2) = 0.15 \) and \( P(a_1) = 0.45 \).
ODDS

3.23. Suppose \( P(E) = p \). The odds that \( E \) occurs is defined to be the ratio \( p : (1 - p) \). Find \( p \) if the odds that \( E \) occurs are \( a \) to \( b \).

Set the ratio \( p : (1 - p) \) to \( a : b \) to obtain

\[
\frac{p}{1 - p} = \frac{a}{b} \quad \text{or} \quad bp = a - ap \quad \text{or} \quad ap + bp = a \quad \text{or} \quad p = \frac{a}{a + b}
\]

3.24. The odds that an event \( E \) occurs is 3 to 2. Find the probability of \( E \).

Let \( p = P(E) \). Set the odds equal to \( p : (1 - p) \) to obtain

\[
\frac{p}{1 - p} = \frac{3}{2} \quad \text{or} \quad 2p = 3 - 3p \quad \text{or} \quad 5p = 3 \quad \text{or} \quad p = \frac{3}{5}
\]

Alternatively, use the formula in Problem 3.21 to directly obtain

\[
p = \frac{a}{a + b} = \frac{3}{3 + 2} = \frac{3}{5}
\]

3.25. Suppose \( P(E) = \frac{5}{12} \). Express the odds that \( E \) occurs in terms of positive integers.

First compute \( 1 - P(E) = \frac{7}{12} \). The odds that \( E \) occurs are

\[
\frac{P(E)}{1 - P(E)} = \frac{\frac{5}{12}}{\frac{7}{12}} = \frac{5}{7}
\]

Thus, the odds are 5 to 7.

UNCOUNTABLE UNIFORM SPACES

3.26. A point is chosen at random inside a circle. Find the probability \( p \) that the point is closer to the center of the circle than to its circumference.

Let \( S \) denote the set of points inside the circle with radius \( r \), and let \( A \) denote the set of points inside the concentric circle with radius \( \frac{1}{2}r \), as pictured in Fig. 3-9(a). Thus, \( A \) consists precisely of those points of \( S \) which are closer to the center than to its circumference. Therefore

\[
p = P(A) = \frac{\text{area of } A}{\text{area of } S} = \frac{\pi (r/2)^2}{\pi r^2} = \frac{1}{4}
\]

![Fig. 3-9](image-url)
3.27. Consider the plane $\mathbb{R}^2$, and let $X$ denote the subset of points with integer coordinates. A coin of radius $1/4$ is tossed randomly on the plane. Find the probability that the coin covers a point of $X$.

Let $S$ denote the set of points inside a square with corners

$$(m, n), \quad (m, n + 1), \quad (m + 1, n), \quad (m + 1, n + 1)$$

where $m$ and $n$ are integers. Let $A$ denote the set of points in $S$ with distance less than $1/4$ from any corner point, as pictured in Fig. 3-9(b). Note that the area of $A$ is equal to the area inside a circle of radius $1/4$. Suppose the center of the coin falls in $S$. Then the coin will cover a point in $X$ if and only if its center falls in $A$. Accordingly,

$$p = \frac{\text{area of } A}{\text{area of } S} = \frac{\pi(1/4)^2}{1} = \frac{\pi}{16} \approx 0.2$$

(Note: We cannot take $S$ to be all of $\mathbb{R}^2$ since the area of $\mathbb{R}^2$ is infinite.)

3.28. On the real line $\mathbb{R}$, points $a$ and $b$ are selected at random such that $0 \leq a \leq 3$ and $-2 \leq b \leq 0$, as shown in Fig. 3-10(a). Find the probability $p$ that the distance between $a$ and $b$ is greater than $3$.

The sample space $S$ consists of the ordered pairs $(a, b)$ and so forms a rectangular region shown in Fig. 3-10(b). On the other hand, the set $A$ of points $(a, b)$ for which $d = a - b > 3$ consists of those points which lie below the line $x - y = 3$, and hence form the shaded region in Fig. 3-10(b). Thus

$$p = P(A) = \frac{\text{area of } A}{\text{area of } S} = \frac{2}{6} = \frac{1}{3}$$

3.29. Three points are selected at random from the circumference of a circle. Find the probability $p$ that the three points lie on a semicircle.
Suppose the length of the circumference is $2s$. Let $x$ denote the clockwise arc length from $a$ to $b$, and let $y$ denote the clockwise arc length from $a$ to $c$, as pictured in Fig. 3-11(a). Thus

$$0 < x < 2s \quad \text{and} \quad 0 < y < 2s \quad (*)$$

Let $S$ denote the set of points in $\mathbb{R}^2$ for which the condition (*) holds. Let $A$ denote the set of points of $S$ for which any of the following conditions hold:

(i) $x, y < s$, \hspace{1cm} (iii) $x < s$ and $y - x > s$,

(ii) $-x, y > s$, \hspace{1cm} (iv) $y < s$ and $x - y > s$.

Then $A$, shaded in Fig. 3-11(b), consists of those points for which $a, b, c$ lie on a semicircle. Thus

$$p = \frac{\text{area of } A}{\text{area of } S} = \frac{3s^2}{4s^2} = \frac{3}{4}$$

Fig. 3-11

3.30. A coin of diameter $1/2$ is tossed randomly onto the plane $\mathbb{R}^2$. Find the probability $p$ that the coin does not intersect any line of the form $x = k$ where $k$ is an integer.

The lines are all vertical, and the distance between adjacent lines is one. Let $S$ denote a horizontal line segment between adjacent lines, say, $x = k$ and $x = k + 1$; and let $A$ denote the points of $S$ which are at least $1/4$ from either end, as pictured in Fig. 3-12. Note that the length of $S$ is 1 and the length of $A$ is $1/2$. Suppose the center of the coin falls in $S$. Then the coin will not intersect the lines if and only if its center falls in $A$. Accordingly

$$p = \frac{\text{length of } A}{\text{length of } S} = \frac{1/2}{1} = \frac{1}{2}$$

Fig. 3-12