CS 237 Fall 2018, Homework 03 -- SOLUTION

Due date: PDF file due Thursday September 27th @ 11:59PM (10% off if up to 24 hours late) in GradeScope

General Instructions

Please complete this notebook by filling in solutions where indicated. Be sure to “Run All” from the Cell menu before submitting.

You may use ordinary ASCII text to write your solutions, or (preferably) LaTeX. A nice introduction to LaTeX in Jupyter notebooks may be found here: http://data-blog.udacity.com/posts/2016/10/latex-primer/

As with previous homeworks, just upload a PDF file of this notebook. Instructions for converting to PDF may be found on the class web page right under the link for homework 1.

Problem 1

How many distinct (identical-looking) arrangements of the word COMMANDING are there if:

(a) There are no restrictions other than the normal one that the two 'M's look the same and the two 'N's look the same?

(b) If you do not change the order of the vowels or consonants?

(c) If you do not change the order of the vowels but consider any order of the consonants?

Hint: Obviously, you will use multinomial coefficients; for (b) and (c) you may use the "unordered principle" (accounting for duplicate letters) or you may think about replacing vowels with the letter "V" (which represents the "slot" where a vowels goes, since if vowels must be in order, once you know the locations of the vowels, you can only put them in the slots in one way); and similarly for consonents (replacing by the letter "C").

Solution:

(a) This is permutation with duplicates, and you must "factor out" the permutations which differ only in the arrangement of the duplicate letters. There are 2 M's and 2 N's, so we have \( \frac{10!}{2! 2!} = 907,200 \).

(b) You should always try to avoid the "brute force" approach of constructing permutations in detail, and try to interprete the problem in terms of the basic principles we have been investigating.

Using the unordering principle, you would take the result from (a) and unorder it -- in this problem you must do this in two ways. There are 3 vowels so \( 3! = 6 \) permutations for the vowels; for the consonants you need to account for the duplicates, so \( \frac{7!}{2! 2!} = 1260 \). So there are \( \frac{907,200}{6 \times 1260} = 120 \) possible arrangements

If you use the idea of replacing consonants and vowels, you are considering the permutations of CVCCVCCVCC. There are \( \frac{10!}{3! 2! 2!} = 120 \) permutations of this pattern.

(c)

Factor the permutation of the 3 vowels and the permutation of the 2 M's and 2 N's

\( \frac{10!}{(3! 2! 2!)} = 151,200 \)

Problem 2
(Circular Permutations) In how many ways can 7 people \{A, B, C, D, E, F, G\} be seated at a round table if

(a) A and B must not sit next to each other;

(b) C, D, and E must sit together (i.e., no other person can sit between any of these three)?

(c) A and B must sit together, but neither can be seated next to C or D.

Consider each of these separately. For (c) you may NOT simply list all possibilities, but must use the basic principles we have developed (you may check your work with a list if you wish).

Hint: Conceptually, think of the groups of two or three people as one "multi-person" entity in the overall circular arrangement. It may help to draw a diagram, fixing a particular person at the top of the circle (thereby eliminating the duplicates due to rotations).

Solution:

(a) We must eliminate all those where the two are seated together (effectively, being one combined "multi-person" among 6), i.e., \(\frac{6!}{5!}\), and then account for the duplication (since the two can be in any order, i.e., are a permutation of 2 people). (Another example of the unordering principle.) So we have \(\frac{6!}{5!} * 2! = 240\) ways that these two can be seated together. So \(720 - 240 = 480\) ways.

(b) The "multi-person" has three people, and so we have \(\frac{3!}{3!} = 4! = 24\) arrangements including the multi-person, and 3! permutations of the three in the group. Thus \(24*6 = 144\) ways.

(c) Lenka's solution

choose two people to sit next to (AB) or (BA), \(C(3, 2) = 3\)

These two people can be arranged in \(2! = 2\) ways

The other three can sit in \(3! = 6\) ways

A, B can sit in \(2! = 2\) ways

\(3 * 2 * 6 * 2 = 72\) ways

Another way of thinking about it:

The multi-person has 2 people, so we have 5 people (including one multi-person). Therefore there are \(6!/6=5!/120\) ways, and 2! permutations of the multi-person. Therefore \(120 * 2 = 240\) without the constraint on C and D.

Now we must eliminate all those where C and D sit next to the multi-person.

First consider where both C and D sit next to the multi-person. There are 2 ways for C and D to sit next to the multi-person, 2! ways for A and B to sit next to each other, and 3! ways for the remaining people to sit, so \(2*2!*3! = 24\) ways

Then, consider where only C or only D sit next to the multi-person. Choose 1 from C or D: \(C(2, 1) = 2\) ways. There are 2! ways for A and B to sit next to each other. There are 2 ways for the the chosen one and the multi-person to sit next to each other. The other one in C, D who doesn't sit next to the multi-person can choose \(C(3, 1) = 3\) positions to sit. There are 3! for the rest of the 3 people to sit. So \(2*2!*2*3!*3! = 144\)

So \(240-24-144 = 72\)

Problem 3
(Permutations and maybe Combinations) Suppose 2 cards are drawn without replacement (the usual situation with cards) from an ordinary
deck of 52 randomly shuffled cards. Find the probability that:

(a) The first card is not a ten of clubs or an ace;

(b) The first card is an ace, but the second is not;

(c) The cards have the same denomination (i.e., both are Aces, both are 2's, both are 3's, etc.);

(d) At least one card is a diamond;

(e) Not more than 1 card is a picture card (Jack, Queen, King).

Solution:

(a) If we remove the aces and the ten of clubs, we have 47 cards left, so 47/52 = 0.9038.

(b) P(first is ace) = 4/52 and P(second not ace) = 48/51. Thus, (4/52)*(48/51) = 192/2652 = 0.0724.

(c) Two ways to do this. Choose a denomination and then choose two of the four of that denomination: C(13,1) * C(4,2) / C(52,2) = 1/17 =
0.0588. OR, simpler, after the first card is chosen, you have to match the denomination with the second card, so you have 3 chances out of
51: 3/51 = 1/17.

(d) As usual, "at least" should tip you off to use the complement. P(no diamonds) = P(first card not a diamond) P(second card not a diamond)
= (39/52)(38/51) = 1482/2652 = 0.5588; so P(at least one diamond) = 0.4412.

(e) There are 4*3 = 12 picture cards (J,Q,K of each suit). The choices are 0, 1, or 2 picture cards; the question asks for 0 or 1, but using the
complement we can derive the answer by just considering 2. P(first a picture card) = 12/52, and P(second a picture card) = 11/51. P(two
picture cards) = 132/2652 = 0.0498; P(not more than 1 picture card) = 0.9502.

Problem 4

(Permutations) We draw 5 cards at random from an ordinary deck of 52 cards (as usual, without replacement). In this case, we will think of the
5 cards as a sequence.

(a) If the first 2 are spades, what is the probability that the remaining 3 are also spades?

(b) If the first 2 cards are spades, what is the probability that among the remaining 3 cards there will be no diamonds?

(c) The first two cards drawn are of different suits.

(d) The last two cards drawn are of different suits.

Hint: For (a) and (b) you can do this with conditional probabilities or not. For (d), ask yourself whether reversing a sequence changes the
probabilities.
Solution:

(a) A: first 2 are spades, B: remaining 3 are spades

\[ P(A) = \frac{13}{52} \cdot \frac{12}{51}, \quad P(AB) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} \cdot \frac{10}{49} \cdot \frac{9}{48} \]

\[ P(B|A) = \frac{P(AB)}{P(A)} = 0.0084 \]

(b) A: first 2 are spades, B: remaining 3 containing no diamonds

\[ P(A) = \frac{13}{52} \cdot \frac{12}{51}, \quad P(AB) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{37}{50} \cdot \frac{36}{49} \cdot \frac{35}{48} \]

\[ P(B|A) = \frac{P(AB)}{P(A)} = 0.3964 \]

(c) Choose a card, compute the prob that the second doesn't have the same suit

\[ \frac{39}{51} = 0.7647 \]

(d) The same as (c)

Problem 5

(Permutations with Duplicates) Let K be the number of distinct birthdays among four persons selected uniformly at random. Assume that all years have 365 days and birthdays are randomly distributed throughout the year.

(a) What is the probability that K = 1?

(b) What is the probability that K = 3?

(c) What is the probability that K = 4?

(d) What is the probability that K = 2?

Here (not all possibilities, just some examples) is what I mean by distinct birthdays:

- "1 distinct birthday": Alice, Bob, Raul, and Yi-Fei were all born on January 1st
- "2 distinct birthdays": Alice and Bob were born on Jan 1st, but Raul and Yi-Fei were born on July 23rd;
- "2 distinct birthdays": Alice was born on Jan 1st, but Bob, Raul, and Yi-Fei were born on July 23rd;
- "3 distinct birthdays": Alice and Bob were born on Jan 1st, Raul was born on July 3rd, and Yi-Fei was born on Dec 4th
- "4 distinct birthdays": All 4 were born on different days.

Hint: It is simpler to think of this as choosing a sequence of k distinct birthdays for k = 1, ..., 4, and then figuring out the number of patterns of shared birthdays; the non-shared birthdays are counted by the original permutation. The messiest one is K = 2, but is trivial if you do the others first!
Solution: It is simpler to think of this as choosing a sequence of k birthdays, and then figuring out the number of patterns of shared birthdays; the non-shared birthdays are counted by the original permutation.

So, we first have

- Number of sequences length 4 of 1 birthday = $P(365,1) = 265$
- Number of sequences length 4 of 2 different birthdays = $P(365,2) = 365 \times 364$
- Number of sequences length 4 of 3 different birthdays = $P(365,3) = 365 \times 364 \times 363$
- Number of sequences length 4 of 4 different birthdays = $P(365,4) = 365 \times 364 \times 363 \times 362$

Now we must figure out the number of ways of arranging the sharing. The best way to do this is to work out the patterns of distinct birthdays and count the number of possible ways these can be taken by the four individuals. This will involve multinomial coefficients, because the shared birthdays are not distinguishable.

For one birthday, we have a single pattern AAAA. For two birthdays, we have two: ABBB, which has $4!/3! = 4$ permutations, and AABB, for which we have $4!/(2!2!) = 3$ permutations, giving a total of 7 in total (note that we do not have to consider AAAB because these will already have been counted in the first pattern); for 3 we have the single pattern AABC which has $4!/2! = 6$ permutations, and for 4 we have the single pattern ABCD.

Thus, we have

- $P(X=1) = \frac{365}{365^4} = 2.056465398684376261 \times 10^{-8}$
- $P(X=2) = \left(\frac{7}{365}\right) \frac{365 \times 364}{365^4} = 5.239873835847797 \times 10^{-5}$
- $P(X=3) = \left(\frac{6}{365}\right) \frac{365 \times 364 \times 363}{365^4} = 0.01630349316353784024862$
- $P(X=4) = \left(\frac{365 \times 364 \times 363 \times 362}{365^4}\right) = 0.983644087533449695...$

These do indeed sum to 1.0, which convinces us that we have accounted for all the possibilities.

Of course, as in the hint, you can eliminate the case $K=3$ by doing the others first, summing, and subtracting from 1.

**Problem 6**

(Combinations and Permutations with Duplicates) In this problem, we have a deck of 52 cards and we shuffle them randomly and deal the whole deck out to 4 people, so that each player has 13 cards. As usual, when dealing cards, it is without replacement and the "hand" that each player has is a set.

(a) What is the probability that each player will receive all cards of one suit (i.e., one gets all clubs, another all hearts, another all diamonds, and another all spades)?

(b) What is the probability that each player will receive exactly three face cards?

Hint: (a) is a simple application of combinations and permutations with duplicates; start with counting the number of ways of assigning suits to the players. (b) is permutation with duplicates; think of dealing out all 52 cards in a sequence, and then giving the first 13 to the first player, the second 13 to the second player, and so on. This is the sample space. Then consider how many such arrangements satisfy the constraint.
Solution:

(a) \( A = \text{Number of ways to deal card to each player: } C(52,13) \times C(39,13) \times C(26,13) \times C(13,13) \)

\( B = \text{Number of ways to assign suit to each play: } C(4,1) \times C(3,1) \times C(2,1) \times C(1,1) \)

\( A/B = 4.4739 \times 10^{28} \)

(b) \( A = \text{Number of ways to deal card to each player: } C(52,13) \times C(39,13) \times C(26,13) \times C(13,13) \)

\( B = \text{Number of ways to deal 10 number cards to each player: } C(40,10) \times C(30,10) \times C(20,10) \times C(10,10) \)

\( C = \text{Number of ways to deal three face cards to each play: } C(12,3) \times C(9,3) \times C(6,3) \times C(3,3) \)

\( A \times B / C = 0.0324 \)

Problem 7

(Combinations -- Poker Probabilities) Suppose you deal a poker hand of 5 cards from a standard deck as discussed in lecture.

(a) What is the probability of a flush (all the same suit) of all red cards?

(b) What is the probability of a full house where the 3-of-a-kind include two black cards, and the 2-of-a-kind are not clubs?

(c) What is the probability of a single pair, where among the 3 non-paired cards, we have 3 distinct suits?

(d) What is the probability of a pair, where 3 of the cards are black and 2 are red?

Hint: These are straight-forward modifications of the formulae given in lecture, except for (d). For that, think about the two different parts of the problem, and observe that they are independent (the colors are independent of the ranks/denominations). You may quote formulae already given in lecture or lab without attribution and without explaining how to get the formula.

Solution

(a) \( A = \text{Number of ways to choose a red suit } = C(2,1) \)

\( B = \text{Number of ways to choose a hand of 5 cards from a red suit } = C(13,5) \)

\( C = \text{Number of ways to choose a hand of 5 cards from a deck } = C(52,5) \)

\( A \times B / C = 0.000099 = 0.0010 \)

(b) \( A = \text{Choose a denomination for the 3-of-a-kind: } C(13,1), \)

Any denomination has 2 red cards and 2 black cards, so there are 2 ways for a 3-of-a-kind to include 2 black cards

\( B = \text{Choose another denomination for the 2-of-a-kind: } C(12,1) \)

For every denomination, we have 3 cards when we exclude Club, so there are \( C = C(3,2) \) ways to get a 2-of-a-kind for that denomination

\( D = \text{Number of ways to choose a hand of 5 cards from a deck } = C(52,5) \)

\( A \times 2 \times B \times C / D = 0.0004 \)

(c) \( A = \text{Choose a denomination for the pair } = C(13,1), \)

\( B = \text{Choose 2 cards from that denomination } = C(4,2) \)

\( C = \text{Choose three different denominations for the non-paired cards: } C(12,3) \)

\( D = \text{Choose three distinct suits for the non-paired cards: } C(4,1) \times C(3,1) \times C(2,1) \text{ or } P(4,2) \)

\( E = \text{Number of ways to choose a hand of 5 cards from a deck } = C(52,5) \)

\( A \times B \times C \times D / E = 0.1585 \)

(d) A = Choose a denomination for the pair, the pair has 1 red 1 black, choose a denomination and red suit for 1 red and and denominations and suits for the 2 black cards. =

$$\binom{13}{1} \cdot \binom{2}{1} \cdot \binom{2}{1} \cdot \binom{12}{1} \cdot \binom{2}{1} \cdot \binom{11}{2} \cdot \binom{2}{1} \cdot \binom{2}{1}$$

B = Choose a denomination for the pair, the pair has 2 red cards, choose 3 denominations, and for each, a black suit:

$$\binom{13}{1} \cdot \binom{2}{2} \cdot \binom{12}{3} \cdot \binom{2}{1} \cdot \binom{2}{1} \cdot \binom{2}{1}$$

C = Choose a denomination for the pair, the pair has 2 black cards, choose 1 black and 2 red for non-paired, as in (A):

$$\binom{13}{1} \cdot \binom{2}{2} \cdot \binom{12}{2} \cdot \binom{11}{1} \cdot \binom{2}{1} \cdot \binom{2}{1}$$

D = Number of ways to choose a hand of 5 cards from a deck = C(52,5)

$$(A+B+C) / D = 0.1409$$

Problem 8

(Combinations) Consider the following problem: “From an ordinary deck of 52 cards, seven cards are drawn at random and without replacement. What is the probability that at least one of the cards is a King?” A student in CS 237 solves this problem as follows: To make sure there is a least one King among the seven cards drawn, first choose a King; there are C(4,1) possibilities; then choose the other six cards from the 51 cards remaining in the deck, for which there are C(51,6) possibilities. Thus, the solution is C(4,1)*C(51,6) / C(52,7) = 0.5385.

However, upon testing the problem experimentally, the student finds that the correct answer is somewhat less, around 0.45.

(a) Calculate the correct answer using the techniques presented in class;

(b) Explain carefully why the student’s solution is incorrect.

Solution:

(a) We use the inverse method: there are C(48,7) seven-card hands with no Kings, so the probability of at least one King is

$$1.0 - \frac{C(48,7)}{C(52,7)} = 0.4496$$

(b) The problem is that hands with more than one King are counted multiple times. For example, { KH, KD, 2H, 5S, JH, 10C, AD } is counted twice: once when the KH is counted as part of the C(4,1) and the KD is counted as part of C(51,6), and again when KD is part of C(4,1) and KH is part of C(51,6). So a hand with n Kings, n > 1, will be counted n times.

Problem 9
(Combinations, Subsets, and Partitions) Suppose you have a committee of 10 people.

(a) How many ways are there to choose a group of 4 people from these 10 if two particular people (say, John and Dave) can not be in the group together?

(b) How many ways are there to choose a team of 5 people from these 10 with one particular person being designated Captain and another particular person being designated Co-Captain?

(c) How many ways are there to separate these 10 people into two groups, if no group can have less than 2 people?

(d) How many ways are there to separate these 10 people into two teams of 6 and 4 people?

(e) How many ways are there to separate these 10 people into four groups of 2, 2, 3, and 3 each?

[Hint: For (c) -- (e), think about whether you are over-counting or not.]

Solution:

For (a), we have to remove from the total in (a) those committees on which John and Dave both sit; but since the only choice is the other two people, this is \( \binom{8}{2} = 28 \), so \( 210 - 28 = 182 \).

(b) The captain and co-captain are a sequence, and the remaining 3 are a set, so \( P(10,2) \cdot \binom{8}{3} = 5040 \)

For (c) and (d), note that if we choose one group, then the other group is determined as well; and so we should consider all possible subsets (one group), which determines the other set. However, this number overcounts by a factor of 2, since we have counted each group and its complement (each of which produces the same pair of groups, just exchanged); e.g., if one group is \{1,2\}, then other group is \{3,4,5,6,7,8,9,10\}; but if we choose \{3,4,5,6,7,8,9,10\}, then the other group is \{1,2\}, and these represent the same separation into two groups. So count the number for one group and divide by two.

Thus, for (c), we can count any group that does not have 0, 1, 9, or 10 people (the latter two would produce a complement with 1 or 0 people). There is only one group with 0, one group with 10, ten groups with 1, and ten groups with 9. The number of subsets of a 10-element set is 210. So we have \( 210 - 22 = 1024 - 22 = 1002 \). Dividing by 2 we get 501 ways.

(d) \( \binom{10}{6} = 210 \)

(e) \( \binom{10}{2} \cdot \binom{8}{2} \cdot \binom{6}{3} \cdot \binom{3}{3} / (2^2) = 6300 \)

**Problem 10**

Among 25 Senate candidates, the 11 (all Republicans) think global warming is a myth, the 8 (all Democrats) believe that global warming is real, and the rest (from the "Spineless Party") have no opinion ("I'm not a scientist"). A newspaper interviews a random sample of 5 of the candidates. What is the probability that

(a) all 5 think global warming is a myth;

(b) all 5 share the same position (i.e., all think it is a myth, all believe it is real, or all have no opinion);

(c) 3 share the same position and 2 share a different position (for example, 3 believe it is a myth and 2 have no opinion).

(d) 2 share the same position, 2 share the same position, but different from the first two, and the remaining candidate has a position different from the other four.

[Hint: Worry about overcounting in (d).]
Solution: There are $C(25,5)$ ways of choosing the five for the interview.

For (a), there are $C(11,5)$ ways of choosing 5 who think it is a myth, so $C(11,5)/C(25,5)=0.0087$.

For (b), there are $C(8,5)$ ways of choosing 5 who think it is real, and $C(6,5)$ ways of choosing 5 who have no opinion. All of these are disjoint, so we can simply add the probabilities: $[C(11,5) + C(8,5) + C(6,5)]/C(25,5) = 0.0097$.

(c) $C(11,3)*C(8,2) + C(11,3)*C(6,2) + C(8,3)*C(11,2) + C(8,3)*C(6,2) + C(6,3)*C(11,2) + C(6,3)*C(8,2) = 12675$

$12675 / C(25,5) = 0.2386$

In detail: there are six different permutations of positions among the groups:

<table>
<thead>
<tr>
<th>majority of 3</th>
<th>minority of 2</th>
<th>Number of possible groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>myth</td>
<td>real</td>
<td>$C(11,3) * C(8,2)$ = 4620</td>
</tr>
<tr>
<td>myth</td>
<td>no opinion</td>
<td>$C(11,3) * C(6,2)$ = 2475</td>
</tr>
<tr>
<td>real</td>
<td>myth</td>
<td>$C(8,3) * C(11,2)$ = 3080</td>
</tr>
<tr>
<td>real</td>
<td>no opinion</td>
<td>$C(8,3) * C(6,2)$ = 840</td>
</tr>
<tr>
<td>no opinion</td>
<td>myth</td>
<td>$C(6,3) * C(11,2)$ = 1100</td>
</tr>
<tr>
<td>no opinion</td>
<td>real</td>
<td>$C(6,3) * C(8,2)$ = 560</td>
</tr>
</tbody>
</table>

Total: 12675

So the probability is $1267/C(25,5) = 0.2386$. Note that you can NOT solve (c) as in the full-house card example (choose a position, then the number of choices, then the second position, and the number of choices for that) because the number of choices is different among the three positions; hence we must explicitly list all possibilities.

(d) $C(11,2)*C(8,2)*C(6,1) + C(11,2)*C(8,1)*C(6,2) + C(11,1)*C(8,2)*C(6,2) = 20460$

$20460 / C(25,5) = 0.3851$