Problem 1. Suppose that each day the price of a stock moves up 1/8th of a point with probability 1/3 and moves down 1/8th of a point with probability 2/3. If the price fluctuations from one day to the next are independent, what is the probability that after six days the stock has its original price? [Hint: Use the binomial.]

Solution: If $S$ is the number of days the stock moves up, and $D$ is the number of days the stock moves down, then the price will be the same after 6 days if and only if $S = D = 3$. Let success = stock moves up; then this is a binomial $B(6,1/3)$ and the probability that the stock stays the same is

$$P(X = 3) = C(6,3) \times (1/3)^3 \times (2/3)^3 = 0.2195.$$ 

Problem 2. Suppose we draw cards at random and with replacement from a standard deck of 52 cards successively until we draw an Ace.

(a) What is the probability that this occurs on the 5th draw?

(b) What is the probability that at least 10 draws are needed?

(c) What is the probability that between 3 and 7 draws (inclusive) are needed?

Solution: This is a RV $X \sim Geo(1/13)$.

(a) $P(X = 5) = (12/13)^4(1/13) = 0.0558$.

(b) $1.0 - [ P(X=1) + P(X=2) + \ldots + P(X=9) ] = 0.4866$.

(c) $P(X=3) + P(X=4) + \ldots + P(X=7) = 0.2810$.

Problem 3. Suppose we do the experiment in the previous problem without replacement; answer all three parts in this new experiment (Hint: This is NOT a Geometric distribution).

Solution: We use a simple Python function to calculate the probabilities:

```python
def p(k):
    # calculate P(X = k) and trace to be sure
    prod = 1
    i = 0
    for i in range(k-1):
        print(str(48-i) + " / " + str(52-i))
        prod *= (48-i)/(52-i)
    print(str(4) + " / " + str(52-k+1))
    prod *= 4/(52-k+1)
    return prod
```


(b) $1.0 - [ p(1) + p(2) + \ldots + p(9) ] = 0.4559$.

(c) $p(3) + p(4) + \ldots + p(7) = 0.3003$.

The following two problems use the Poisson. Remember that one of the key features of a Poisson Process is that it has a mean arrival rate $\lambda = \text{number of arrivals} / \text{unit time}$. This rate scales linearly, so that the mean number of arrival in 10 unit times is $10\lambda$ and the number of arrivals in half a unit time is $\lambda/2$. 
Problem 4. The number of wrong-number calls received per day in the CS department is 3; assume rate is constant for any day, including non business days (Saturday and Sunday). What is the probability that they will receive

(a) 2 wrong-number calls tomorrow?

(b) Between 12 and 20 (inclusive) wrong-number calls next week?

(c) At least 3 wrong-number calls during lunch-time (12-1pm) during business hours (M - F) next week?

Solution: This is a straight-forward application of the formula for the Poisson Distribution with \( \lambda = 3 \) and various periods of time. The follow Python code helps with the computations.

```python
def poisson(lamb,k):
    return math.exp(-lamb)*(lamb**k)/math.factorial(k)

# P(lo <= k < hi)
def poissonRange(lamb,lo,hi):
    sum = 0
    for k in range(lo,hi):
        sum += poisson(lamb,k)
    return sum
```

(a) \( P(X=2) = (3^2 * e^{-3}) / 2! = 0.2240 \)

(b) Now \( \lambda = 3*7 = 21 \), so \( P(12 \leq X \leq 20) = \sum_{k=12}^{20} (21^k * e^{-21}) / k! = 0.4581 \)

(c) This is a randomly-chosen (compound) interval of 5 hours, so \( \lambda = 3*5/24 = 0.625 \), and we have \( 1.0 - (0.625^0 * e^{-0.625}) / 0! - (0.625^1 * e^{-0.625}) / 1! - (0.625^2* e^{-0.625}) / 2! = 0.0257. \)

Problem 5. Suppose that in a certain region of California, earthquakes occur at the average rate of 7 per year.

(a) What is the probability of no earthquakes in a particular year?

(b) Suppose a year ago, there were 8 earthquakes in this region, and this year again there were 8; what is the probability that next year there will be at least 8 earthquakes?

(c) What is the probability that in exactly three of the next eight years, no earthquakes occur? [ Hint: First use the Poisson to figure out the probability of no earthquakes in a given year, then use the Binomial.]

Solution: This is obviously \( X \sim \text{Poi}(7). \)

(a) \( P(X = 0) = (7^0 * e^{-7}) / 0! = 0.0009. \)

(b) Trick question! The previous years are irrelevant. \( P(X \geq 8) = 1.0 - \sum_{k=0}^{7} (7^k * e^{-7}) / k! = 0.5987. \)

(c) Use the answer from (a) for \( P(X=0) \); then we have \( Y \sim \text{B}(8,0.0009) \), so \( P(Y=3) = \text{C}(8,3) * 0.0009^3*0.9991^5 = 4 x 10^{-8}. \)

Problem 6. In this problem we will compare the Binomial and the Poisson as an approximation to the Binomial. The probability that a patient will have a bad reaction to a new drug being tested is 0.001. The new drug is administered to 2000 people.

(a) What is the probability that exactly 3 individuals will develop a bad reaction (use the Binomial);

(b) What is the probability that least 2 individuals will develop a bad reaction (use the Binomial);
c) Redo (a) but using the Poisson, letting $\lambda$ = the expected value of the Binomial;

(d) Redo (b) using the Poisson.

Solution:

(a) \( X \sim B(2000,0.001) \) \quad \( P(X=3) = \binom{2000}{3} \times 0.001^3 \times 0.999^{1997} = 0.1805 \)

(b) \( P(X\geq2) = 1.0 - P(X=0) - P(X=1) = 0.5941 \)

(c) \( X \sim Poi(2000\times0.001) \) \quad \( P(X=3) = e^{(-2)} \times 2^3 / 3! = 0.1804 \)

(d) \( P(X\geq2) = 1.0 - P(X=0) - P(X=1) = 0.5940 \)

Problem 7. This problem again compares the Binomial and the Poisson. Suppose in the previous problem, the probability that the patient will develop a bad reaction is 0.3 and the number of patients is 200.

(a) What is the probability that precisely 60 individuals will develop a bad reaction? Use the Binomial.

(b) What is the probability that at most individuals 10 will develop a bad reaction? Use the Binomial.

(c) Repeat (a) using the Poisson.

(d) Repeat (b) using the Poisson.

Solution:

(a) \( X \sim B(200,0.3) \) \quad \( P(X=60) = \binom{200}{60} \times 0.3^{60} \times 0.7^{140} = 0.0615 \)

(b) \( X \sim B(200,0.3) \) \quad \( P(X\leq10) = \sum_{x=0}^{10} \binom{200}{x} \times 0.3^{x} \times 0.7^{200-x} = 5.582 \times 10^{-19} \)

(c) \( X \sim Poi(200\times0.3) \) \quad \( P(X=60) = e^{(-60)} \times 60^{60} / 60! = 0.0514 \)

(d) \( X \sim Poi(200\times0.3) \) \quad \( P(X\leq10) = \sum_{x=0}^{10} e^{(-60)} \times 60^{x} / x! = 1.744 \times 10^{-15} \)

For (a) and (c), the Poisson is about 16\% too low; for (b) and (d) the estimate is again too low, this time by a factor of \( 1 / 3124 \). Not a good approximation at all!

The remaining problems have to do with the Exponential Distribution, which will be covered on Tuesday.

Problem 8. Let \( X \) be a continuous random variable with a PDF of the form

\[
\begin{align*}
    f(x) & = \frac{x}{4} \quad \text{if } 1 \leq x \leq 3 \\
    & = 0 \quad \text{otherwise}
\end{align*}
\]

graphed as follows:
(a) Determine the formula for the CDF $F(x)$ using geometrical techniques.

(b) Determine the formula for the CDF $F(x)$ using an integral.

(c) Sketch or plot the CDF $F(x)$

(d) Determine $P(X \geq 2)$

(e) Determine $E(X)$ (use the integral instead of the discrete summation).

**Solution:**

(a) The figure is composed of a rectangle of height 0.25 and width 2.0, plus a triangle of height 0.75 and width 2.0. Doing these separately, at a point $1 \leq x \leq 3$, we have a rectangle of area $(x-1)$ and height 0.25, so its area is $(x-1)/4$; we also have a triangle of base $(x-1)$ and height $(x-1)/4$ (since the slope of the diagonal line is 1/4), whose area is $(x-1)^2/8$. So we have

$$
\frac{x-1}{4} + \frac{(x-1)^2}{8} = \frac{2(x-1) + (x-1)^2}{8} = \frac{x^2-2}{8}.
$$

(b) The indefinite integral of $f(x)$ is:

\[\int_{x/4} x \, dx = \frac{x^2}{8} + \text{constant}\]

and so

\[F(x) = \begin{cases} 0 & \text{for } x < 1 \\ x^2/8 - 1/8 & \text{for } 1 \leq x \leq 3 \\ 1 & \text{for } x > 3 \end{cases}\]
(d) Determine \( P(X \geq 2) = 1 - P(X < 2) = 1 - F(2) = 1 - \left(\frac{2^2}{8} - \frac{1}{8}\right) = 1 - \left(\frac{1}{2} - \frac{1}{8}\right) = \frac{5}{8} = 0.625 \).

(e) 
\[
E[X] = \int_1^3 \frac{x^2}{4} \, dx = \frac{x^3}{12} \bigg|_1^3 = \frac{27}{12} - \frac{1}{12} = \frac{26}{12} = \frac{13}{6},
\]

\[= 2.1667.\]

**Problem 9.** (Exponential Distribution) Suppose that every three months, on average, an earthquake occurs in California.

(a) What is the probability that an earthquake will not occur for another year?

(b) What is the probability that the next earthquake occurs after 3 but before 7 months?

**Solution:** This is \( X \sim \text{Exp}(1/3) \), where \( \lambda = 1 \text{ quake} / 3 \text{ months} \)

(a) \( P(X > 12) = e^{-\frac{12}{3}} = 0.0183 \).

(b) \( P(X > 3) - P(X > 7) = e^{-1} - e^{-\frac{7}{3}} = 0.2709 \).

**Problem 10.** (Exponential Distribution) Guests arrive at a hotel, in accordance with a Poisson Process, at a rate of 5 per hour. Suppose that for the last 10 minutes no guest has arrived. What is the probability that

(a) The next one will arrive in less than 2 minutes?

(b) From the arrival of the 10th to the arrival of the 11th guest takes no more than 2 minutes.

**Solution:** This is \( X \sim \text{Exp}(1/12) \), where \( \lambda = 1 \text{ arrival} / 12 \text{ minutes} \). The past has no relevance (ha!). So:

(a) \( P(X < 2) = 1.0 - e^{-\frac{2}{12}} = 0.1535 \).

(b) Same!