 Lecture 7:

- Poker Probability: finish with analysis of standard poker hands
- Counting subsets
- Powerset and its relationship to $\binom{N}{K}$
- Counting Partitions and the unordering principle!

Next Time: Discrete Random Variables, Read Chapter 5.1 – 5.3
A poker hand is 5 cards (a set) chosen without replacement.

How many possible poker hands?

$$|S| = \binom{52}{5} = 2,598,960$$
Poker Probability

Poker hands are a great example of how to think about probability involving sets.

**Example:** What is the probability that you get a hand with 3 red cards and 2 black cards?

(Hint: Construct the hand by calculating how many such hands are possible, by constructing independent parts of the hand, and multiplying....)
Poker Probability

Poker hands are a great example of how to think about probability involving sets.

Example: What is the probability that you get a hand with 3 red cards and 2 black cards?

Solution:

\[
\frac{\binom{26}{3} \binom{26}{2}}{\binom{52}{5}} = 0.3251
\]

By the way, Wolfram Alpha is the way to go when doing these problems...
Poker Probability

Poker hands are a great example of how to think about probability involving sets.

**Example:** What is the probability that you get a hand with 3 red cards and 2 black cards?

**Digression:** Notice that you can also calculate this using sequences and permutations, but it is a bit more complicated, and you have more opportunities to get something wrong...

What is the probability of the exact sequence $\text{R R R B B}$?

$$\frac{26}{52} \times \frac{25}{51} \times \frac{24}{50} \times \frac{26}{49} \times \frac{25}{48} = 0.0325$$

Now unorder it! How many permutations of this sequence of 5 symbols with duplicates?

$$\frac{5!}{3! 2!} = \frac{120}{6 \times 2} = 10$$

$$0.0325 \times 10 = 0.3251$$
Problem: What is the probability that a five-card hand has at least 3 Diamonds?

Solution: You need to separate this problem into cases, and might as well choose 3, 4, or 5 Diamonds, and for each find the probability and sum:

\[ P(3 \text{ Diamonds}) = \frac{\binom{13}{3} \binom{39}{2}}{\binom{52}{5}} = 0.0815 \]

\[ P(4 \text{ Diamonds}) = \frac{\binom{13}{4} \binom{39}{1}}{\binom{52}{5}} = 0.0107 \]

\[ P(5 \text{ Diamonds}) = \frac{\binom{13}{5}}{\binom{52}{5}} = 0.0005 \]

These sum to 0.0928.
Poker Probability

**Problem:** What is the probability of a Flush (all the same suit)?

**Solution:** Choose a suit and then choose 5 cards from that suit:

\[
\frac{\binom{4}{1} \binom{13}{5}}{\binom{52}{5}} = 0.00198079
\]

Note: This is the cumulative probability, in that all the cards are the same suit, and includes the straight and royal flushes. If we wish to exclude them, we must subtract all 40 of them:

\[
\frac{\binom{4}{1} \binom{13}{5} - \binom{10}{1} \binom{4}{1}}{\binom{52}{5}} = 0.00196540
\]
Poker Probability

Problem: What is the probability of a Straight?

Solution: Assuming Ace is above King, there are 10 sequences which form a straight, so just choose one of the 10 and then the random suits:

\[
\frac{10 \times 4^5}{52 \choose 5} = 0.00394
\]

\[
\frac{10 \times 4^5 - 10 \times 4 \times {4 \choose 4} \times {52 \choose 5}}{52 \choose 5} = 0.003925
\]
Poker Probability

**Problem:** What is the probability of a Pair, 3-of-a-Kind, and 4-of-a-Kind?

**Solution:** First choose the denomination of the 2, 3 or 4, then those suits, then the remaining cards of different denominations:

**Pair:**

\[
\frac{\binom{13}{1} \binom{4}{2} \binom{12}{3} \binom{4}{1}^3}{\binom{52}{5}} = 0.4226
\]

**3-of-a-Kind:**

\[
\frac{\binom{13}{1} \binom{4}{3} \binom{12}{2} \binom{4}{1}^2}{\binom{52}{5}} = 0.0211
\]

**4-of-a-Kind:**

\[
\frac{\binom{13}{1} \binom{4}{4} \binom{12}{1} \binom{4}{1}}{\binom{52}{5}} = 0.00024
\]
Poker Probability

**Problem:** What is the probability of a Full House (3 of one denomination and 2 of another)?

**Solution:** First choose the denomination of the 3, then those 3 suits, then the denomination of the pair, then those 2 suits:

\[
\frac{\binom{13}{1} \binom{4}{3} \binom{12}{1} \binom{4}{2}}{\binom{52}{5}} = 0.00144
\]
Counting Sets: Power Set

The Power Set of a set $S$ is the set of all subsets ( = set of all events ):

$$\mathcal{P}(S) \overset{\text{def}}{=} \{ A \mid A \subseteq S \}$$

The cardinality of Power Set:

$$|\mathcal{P}(S)| = 2^{|S|}$$

This is easy to see if we consider the enumeration of all sequences of $\{ T, F \}$ of length $|S|$, stating which elements of $S$ are in the subset:

$$\{ x, y \}$$

$$\begin{array}{c}
 x & y & z \\
 T & T & F \\
\end{array}$$

$$|\{ T, F \}|^{|S|} = 2^{|S|}$$
Counting Sets: Power Set and Combinations

There is of course a strong connection between the power set and combinations.

\[ |\mathcal{P}(S)| = 2^{|S|} = \sum_{k=0}^{|S|} \binom{|S|}{k} \]
Counting Sets: Power Set and Combinations

**Problem:** A pizza shop claims they serve “more than 1000 kinds of pizza.” You investigate and find they offer 10 different toppings (including cheese and tomato sauce among the 10). Is their claim correct? What about if we insist that a pizza must have cheese and tomato sauce at the very least?

**Solution:** Technically, yes, if you include all possible combinations of toppings, including cheese or no cheese and tomato sauce or no tomato sauce:

\[ 2^{10} = 1024 \]

But this is a little funny, as it includes the empty set (no toppings, just bare crust!).

If you insist that “pizza” must have cheese and tomato sauce, then we have only

\[ 2^8 = 256 \]
Counting Sets: Partitions

A partition of a set $S$ is a set of disjoint subsets which include every member of $S$:

$$S = \{ 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 \}$$

Partitions:  
$$\{ \{1, 2, 3, 4\}, \{5, 6, 7, 8, 9, 10\} \}$$  
$$\{ \{1\}, \{2, 4\}, \{3, 8\}, \{5, 7, 9\}, \{10\} \}$$  
$$\{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\} \}$$

Not so much:  
$$\{ \{1, 2, 3, 4\}, \{5, 6, 7, 8, 9\} \}$$  Doesn’t have the 10 !  
$$\{ \{1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\} \}$$  Not disjoint !
Counting Sets: Partitions

Counting partitions, let’s try some examples...

**Problem.** Suppose we have five students \( \{ A, B, C, D, E \} \)

We want to divide them into two teams of 3 and 2 people each. How many ways can we do this?

\[
\binom{5}{3} = 10
\]

Note: Once we have chosen the team of 3, the other team is determined!

<table>
<thead>
<tr>
<th>Team of 3</th>
<th>Team of 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B C</td>
<td>D E</td>
</tr>
<tr>
<td>A B D</td>
<td>C E</td>
</tr>
<tr>
<td>A B E</td>
<td>C D</td>
</tr>
<tr>
<td>A C D</td>
<td>B E</td>
</tr>
<tr>
<td>A C E</td>
<td>B D</td>
</tr>
<tr>
<td>A D E</td>
<td>B C</td>
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<tr>
<td>B C E</td>
<td>A D</td>
</tr>
<tr>
<td>B D E</td>
<td>A C</td>
</tr>
<tr>
<td>C D E</td>
<td>A B</td>
</tr>
</tbody>
</table>
Counting Sets: Partitions

Now let’s try 2 teams of 2:

Problem. Suppose we have four students \{ A, B, C, D \}

We want to divide them into two teams of 2 people each. How many ways can we do this?

\[
\binom{4}{2} = 6
\]

Note: Once we have chosen the team of 2, the other team is determined!

Team of 2 Team of 2

A B C D
A C B D
A D B C
B C A D
B D A C
C D A B

Not correct! We have twice as many ways. Since we can’t distinguish between the two teams, there is overcounting!

Same WAY of dividing into 2 teams!
Counting Sets: Partitions

Now let’s try 3 teams of 2:

Problem. Suppose we have six students \{A, B, C, D, E, F\}

We want to divide them into 3 teams of 2 people each. How many ways can we do this?

\[
\binom{6}{2} \binom{4}{2} = 15 \times 6 = 90
\]

Team of 2  Team of 2  Team of 2

<table>
<thead>
<tr>
<th>A B</th>
<th>C D</th>
<th>E F</th>
</tr>
</thead>
<tbody>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>A B</td>
<td>E F</td>
<td>C D</td>
</tr>
<tr>
<td>C D</td>
<td>A B</td>
<td>E F</td>
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<td>...</td>
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<td>...</td>
</tr>
<tr>
<td>C D</td>
<td>E F</td>
<td>A B</td>
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<tr>
<td>EF</td>
<td>A B</td>
<td>CD</td>
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<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>EF</td>
<td>C D</td>
<td>A B</td>
</tr>
</tbody>
</table>

Note: Once we have chosen the first 2 teams of 2, the last team is determined!

All these “ways” of dividing into 3 teams of equal size are the same!

Overcounting by P(3,3) = 3!, correct answer is:

\[
\frac{\binom{6}{2} \binom{4}{2}}{3!} = \frac{90}{6} = 15
\]

The Unordering Principle returns!
Counting Sets: Partitions

**Problem.** Suppose we have 15 students and want to divide them into 2 teams of 3, 4 teams of 2, and a single student who will be referee. How many ways of doing this are there?

**Solution:** Use multi-nomial coefficients to remove the duplicates among teams you can’t distinguish by size:

\[
\frac{\binom{15}{3} \binom{12}{3} \binom{9}{2} \binom{7}{2} \binom{5}{2} \binom{3}{2}}{2! \times 4!} = \frac{2,270,268,000}{48} = 47,297,250
\]
Counting Sets: Partitions

Now suppose we distinguish the teams by NAME.

**Problem.** Suppose we have four students \{ A, B, C, D \}

We want to divide them into two teams of 2 people each called ”Attackers” and “Defenders.” How many ways can we do this?

\[
\binom{4}{2} = 6
\]

<table>
<thead>
<tr>
<th>Attackers</th>
<th>Defenders</th>
</tr>
</thead>
<tbody>
<tr>
<td>A B</td>
<td>C D</td>
</tr>
<tr>
<td>A C</td>
<td>B D</td>
</tr>
<tr>
<td>A D</td>
<td>B C</td>
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<tr>
<td>B C</td>
<td>A D</td>
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<td>A C</td>
</tr>
<tr>
<td>C D</td>
<td>A B</td>
</tr>
</tbody>
</table>

Now there is no overcounting! Switching attackers and defenders gives you a different way. There are no duplicate ways.
Counting Sets: Partitions

**Problem.** Suppose we have 15 students and want to divide them into

-- 2 teams of 3 named “MIT Attackers” and “Harvard Attackers”
-- 4 teams of 2 all defenders (all unnamed); and
-- a single student who will be referee.

How many ways of doing this are there?

**Solution:** Use multinomial coefficients to remove the duplicates among teams you can’t distinguish by size or name:

\[
\frac{\binom{15}{3} \binom{12}{3} \binom{9}{2} \binom{7}{2} \binom{5}{2} \binom{3}{2}}{4!} = \frac{2,270,268,000}{24} = 94,594,500
\]
Problem: What is the probability of Two Pair (2 of one denomination and 2 of different denomination)?

Solution: First choose the denomination of the first pair, then those 2 suits, then the denomination of the pair, then those 2 suits, then the remaining card:

\[
\frac{\binom{13}{1}\binom{4}{2}\binom{12}{1}\binom{4}{2}\binom{11}{1}\binom{4}{1}}{\binom{52}{5}} = 0.09508
\]

But wait.... This doesn’t correspond to the web page OR our experiments, which seem to suggest it is too high by a factor of 2. What is wrong?
Poker Probability One Last Time:

Just another example of you-know-what, in this case, overcounting the two pairs:

\[
\begin{align*}
2D & \quad 2H & \quad 3C & \quad 3D & \quad 5S \\
3C & \quad 3D & \quad 2D & \quad 2H & \quad 5S
\end{align*}
\]

These are the same hand, but would be counted twice!

So we simply have to divide by 2 to get the right number!

\[
\frac{\left(\begin{array}{c} 13 \\ 1 \end{array}\right) \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \left(\begin{array}{c} 12 \\ 1 \end{array}\right) \left(\begin{array}{c} 4 \\ 2 \end{array}\right) \left(\begin{array}{c} 11 \\ 1 \end{array}\right) \left(\begin{array}{c} 4 \\ 1 \end{array}\right)}{2 \times \left(\begin{array}{c} 52 \\ 5 \end{array}\right)} = 0.04754
\]

Same problem as: you have 52 students, and want to select 2 teams of 2, plus a referee.