Lecture 16:

- Review: Poisson Process
- Poisson (Discrete) Distribution
- Poisson as Approximation of Binomial
- Poisson and Exponential: Two ways of looking at Poisson Processes
Review: Exponential Distribution

**Exponential Distribution: Exp(\(\lambda\))**

**Motivation:** If we have a process in which events arrive (hence, 
the Exponential characterizes the inter-arrival time, e.g., "how lon

**Definition:** \(X \sim \text{Exp}(\lambda)\) if

\[
R_{\text{ng}}(X) = [0, \infty) \\
f(t) = \lambda e^{-\lambda t} \\
F(t) = 1.0 - e^{-\lambda t}
\]

\[
E(X) = \frac{1}{\lambda} \\
Var(X) = \frac{1}{\lambda^2}
\]

\[
P(X > t) = e^{-\lambda t} \\
P(X \leq t) = 1.0 - e^{-\lambda t}
\]

where \(e = 2.71828183 \ldots \) (Euler's constant).

**Geometrical Distribution: Geometric(p)**

**Motivation:** This counts the number of Bernoulli trials until the first success occurs.

It can be viewed as a countable sequence of i.i.d. Bernoulli trials:

\(X_1, X_2, X_3, \ldots\)

where we return the smallest index \(i\) for which \(X_i = 1\).

**Definition:** \(X \sim \text{Geometric}(p)\) if

\[
R_X = \{1, 2, \ldots\} \\
P_X(k) = (1 - p)^{k-1} p
\]

**Useful Formulae:**

\[
E(X) = \frac{1}{p} \\
Var(X) = \frac{1 - p}{p^2}
\]

\[
P(X > k) = (1 - p)^k \\
P(X \leq k) = 1.0 - (1 - p)^k
\]
**Poisson Process**

The Poisson Process concept captures an important way of thinking about events randomly occurring through time (or space)... Two things to remember are

- **Events are discrete** (they happen or they don’t – you can think of it as a Bernoulli trial with an outcome of success or failure), but

- **Time and space are continuous**..... the random behavior here is the time of an event.

When an event has happened we say it has arrived. You can think of this as a sequence of real numbers giving the arrival time of an event:

\[
\text{Arrival Times} = \{ 0.4324..., 0.734, 1.389..., 1.453..., 2.1546..., \ldots \}
\]
Poisson Process

Formally, we have the following definition: suppose we have discrete events occurring through time as just described, and let us define a **Counting Random Variable**

\[
N[s..t] = \text{the number of events arriving in the time interval } [s..t]
\]

such that

1) The expected value of \( N[s..t] \):
   a) is a fixed constant \( \lambda \) over any unit interval anywhere in the sequence, and
   b) is proportional to the length \( (t - s) \) of the interval; in particular, for any two non-overlapping intervals of the same length, the expected number of occurrences in each is the same;

2) The number of arrivals in two non-overlapping intervals is independent; and

3) The probability of two events occurring at the same time is 0.

Then this random process is said to be a **Poisson Process**.
Poisson Random Variables (Discrete)

Suppose we have a Poisson Process and we fix the unit time interval we consider (say, 1 second or 1 year, etc.), where the mean number of arrivals in a unit interval is \( \lambda \), and then each time we “poke” the random variable \( X \) we return \( N[0..1), N[1..2), N[2..3), \ldots \), etc.

Then we call \( X \) a Poisson Random Variable with rate parameter \( \lambda \), denoted

\[
X \sim \text{Poi}(\lambda)
\]

where

\[
R_X = \{ 0, 1, 2, 3, \ldots \}
\]

\[
f_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}
\]
Poisson Random Variables

Examples

Assume that arrivals of email in my Inbox are a Poisson Process with rate $\lambda = 10$ messages per hour. Then $X \sim \text{Poi}(10)$ returns the random number of emails which arrive within any particular hour.

What is the probability that I get no emails in the next hour?

$$P(X = 0) = f_X(0) = \frac{e^{-10} \lambda^0}{0!} = e^{-10} = 4.54 \times 10^{-5}$$
Poisson Random Variables

Examples

Assume that arrivals of email in my Inbox are a Poisson Process with rate $\lambda = 10$ messages per hour. Then $X \sim \text{Poi}(10)$ returns the random number of emails which arrive within any particular hour.

What is the probability that I get exactly 10 emails in the next hour?

$$P(X = 10) = f_X(10) = \frac{e^{-10} \lambda^{10}}{10!} = 0.1251$$
Poisson Random Variables

Examples

Unfortunately there is no way to compute the CDF or ranges except by simply adding together all the individual values.

What is the probability that I get between 5 and 15 emails (inclusive) emails in the next hour?

\[ P(5 \leq X \leq 15) = \sum_{k=5}^{15} \frac{e^{-10} 10^k}{k!} = 0.922 \]

\[ X \sim \text{Poi}(10) \]

\[ R_X = \{ 0, 1, 2, 3, \ldots \} \]

\[ P(X = k) = f_X(k) = \frac{e^{-10} 10^k}{k!} \]
Poisson Random Variables

Examples

Note that by the definition of Poisson Processes, “The expected value of $N[s..t]…$ is proportional to the length $(t - s)$ of the interval.”

This means that you can scale a Poisson process to a different time period by simply multiplying or dividing the rate parameter:

Example: If $\lambda = 10$ emails per hour, what is the probability I get no emails in the next hour?

$$P(X = 0) = f_X(0) = \frac{e^{-10} \lambda^0}{0!} = e^{-10} = 4.54 \times 10^{-5}$$

What is the probability I get no emails in the next 15 minutes? Scale the rate to this new period: $\lambda = 10$ emails per hour is equivalent to $\lambda = 2.5$ emails per quarter hour:

$$P(X = 0) = \frac{e^{-2.5} 2.5^0}{0!} = 0.0821$$

Check: Probability of 4 independent quarter-hours in a row = $0.0821^4 = 4.54 \times 10^{-5}$. 
Relationship between Poisson and Exponential

Suppose we have a Poisson Process and we fix the unit time interval we consider (say, 1 second or 1 year, etc.), where the mean number of arrivals in a unit interval is \( \lambda \), and then each time we “poke” the random variable \( X \) we return \( N[0..1] \), \( N[1..2] \), \( N[2..3] \), etc.

Then we call \( X \) a Poisson Random Variable with rate parameter \( \lambda \), denoted

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R_X = \{ 0, 1, 2, 3, \ldots \}
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f_X(k) = \frac{e^{-\lambda} \lambda^k}{k!}
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Relationship between Poisson and Exponential

Suppose we have a Poisson Process, and instead of counting the number of arrivals in each unit interval, we look at the interarrival times, i.e., the amount of time between each arrival.

Intuitively, this is a natural thing to think about: **How long before the next event?**

Let’s define the random variable $Y = “the\ arrival\ time\ of\ the\ first\ event.”$
**Relationship between Poisson and Exponential**

Suppose we have a Poisson Process, and instead of counting the number of arrivals in each unit interval, we look at the interarrival times, i.e., the amount of time between each arrival.

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Let’s define the random variable \( Y = \) “the arrival time of the first event.”

In fact, because the arrivals are independent, at any time \( t \), probabilistically the Poisson process starts all over again (the events don’t remember the past!), so in fact:

\[ Y = \) “the interarrival time between any two events”

Now the question is: **What is the distribution of \( Y \)?**
What is the distribution of $Y$? Since

$$\lambda = E( N[0..1] )$$

and the number of arrivals in an interval is proportional to its length, that is, $E( N[0..2] ) = 2 \cdot E( N[0..1] )$, etc., then $\lambda \cdot t = E( N[0..t] )$

and so the probability that there are exactly $n$ arrivals by time $t$ is

$$P( N[0..t] = n ) = \frac{e^{-\lambda t} (\lambda t)^n}{n!}$$

and

$$P( Y > t ) = P( N[0..t] = 0 ) = \frac{e^{-\lambda t} (\lambda t)^0}{0!} = e^{-\lambda t}$$

$$P( Y \leq t ) = 1 - e^{-\lambda t}$$
What is the distribution of $Y$?

$$P(Y \leq t) = 1 - e^{-\lambda t}$$

Now, this is the CDF of the Exponential, that is,

$$F(t) = \begin{cases} 
1 - e^{-\lambda t} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases}$$

and so if we take the derivative $0 - (-\lambda)e^{-\lambda t} = \lambda e^{-\lambda t}$ we get the PDF:

$$f(t) = F'(t) = \begin{cases} 
\lambda e^{-\lambda t} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases}$$
Optional: The Waiting Time Paradox: