Lecture 5:

- Independence reviewed; Bayes' Rule
- Counting principles and combinatorics;
  - Counting considered as sampling and constructing outcomes; selection with and without replacement;
  - Counting sequences:
    - Enumerations and Cross-products;
    - Permutations;
    - K-Permutations
    - Permutations with Duplicates
    - Circular Permutations
Conditional Probability and Independence: Review

Recall: (Conditional Probability)

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

We can think of this as *conditioning* the sample space:
Independence and Dependence: Review

Recall: (Independent Events)

We say that two events $A$ and $B$ are independent if

$$P(A \mid B) = P(A)$$

or, equivalently, and most importantly as we go forward:

$$P(A \cap B) = P(A) \times P(B)$$

Example:

What is the probability of getting HHT when flipping three fair coins?

$$P(\text{HHT}) = P(H) \times P(H) \times P(T) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 1/8.$$ 

Note: Independence does not depend on physical independence, but on the mathematical relationship. However, it may indicate a lack of causality.
Independence and Dependence: Review

How does this relate to tree diagrams?

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

\( P(A \mid B) \) considers an event B followed by an event A, and how the occurrence of B affects the occurrence of A. What are the labels on a tree diagram of this random experiment?

B occurs (or not)       A occurs (or not)

\[ P(B) \]

\[ P(A \mid B) \]

\[ P(A \cap B) = P(A \mid B) \times P(B) \]

\[ P(A^c \mid B) = 1 - P(A \mid B) \]

\[ P(A^c \cap B) \]

\[ P(A \mid B^c) \]

\[ P(A \cap B^c) \]

\[ P(A^c \mid B^c) \]

\[ P(A^c \cap B^c) \]
Independence and Dependence: Review

When the events are independent, then we have the familiar tree diagram in which we simply write the probabilities of the events on each arc:

B occurs (or not)    A occurs (or not)

\[ P(B) \]

\[ P(A) \]

\[ P(A^c) \]

\[ P(A^c) \]

\[ P(A^c \cap B^c) \]

\[ P(A \cap B) \]

\[ P(A^c \cap B) \]

\[ P(A \cap B^c) \]

\[ P(A^c \cap B) \]

\[ P(A^c \cap B^c) \]

\[ P(A) \]

\[ P(A^c) \]

\[ P(A^c) = 1 - P(A) \]

\[ P(A) = 1 - P(A^c) \]

\[ P(A \cap B) = P(A) \times P(B) \]
Bayes’ Rule

We can rearrange the conditional probability rule in a way that makes the sequence of the events irrelevant -- which happened first, A or B? Or did they happen at the same time? Does it matter?

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B \mid A) = \frac{P(B \cap A)}{P(A)} \]

We can do a little algebra to define conditional probabilities in terms of each other:

\[ P(B \mid A) \times P(A) = P(B \cap A) = P(A \mid B) \times P(B) \]

so:

\[ P(B \mid A) = \frac{P(A \mid B) \times P(B)}{P(A)} \]
Bayes’ Rule

The best way to understand this is to view it with a tree diagram!

\[ P(B | A) = \text{the probability that when } A \text{ happens, it was “preceeded” by } B: \]

If \( A \) has happened, what is the probability that it did so on the path where \( B \) also occurred?

Note:

\[ A = P( A \cap B ) \cup P( A \cap B^c ) \]

So what percentage of \( A \) is due to \( A \cap B \)?

Same calculation as:

\[
P(B | A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}
\]
Bayes’ Rule

This has an interesting flavor, because we can ask about causes of outcomes:

**A Priori Reasoning** -- “I randomly choose a person and observe that he is male; what the probability that it is a smoker?”

“The first toss of a pair of dice is a 5; what is the probability that the total is greater than 8?”

**A Posteriori Reasoning** -- “I find a cigarette butt on the ground, what is the probability that it was left by a man?”

“The total of a pair of thrown dice is greater than 8; what is the probability that the first toss was a 5?”

This seems odd, because instead of reasoning forward from “causes to effects” we are reasoning backwards from “effects to causes” but really it is just different ways of phrasing the mathematical formulae. Time is not really relevant!
Finite Combinatorics

Recall the rule for finite, equiprobable probability spaces:

\[ P(A) = \frac{|A|}{|S|} \]

To work with this definition, we will need to calculate the number of elements in \( A \) and \( S \) and we will analyze this according to how we “constructed” the sample points in \( S \) and in \( A \) during the random experiment.

Compare with how we “construct” the sample space using a tree diagram!
Finite Combinatorics

The way in which we “construct” the sample space almost always follows what we might characterize as a sampling process:

Collection of N Basis Objects (with or without duplicates)

......

Selection of K Objects (with or without replacement)

\{ _, _, ......, _ \} set (unordered)

or

\[ _, _, ......, _ \] sequence (ordered)
Finite Combinatorics

(A) There is a basis collection of $N$ elements that are used to create an outcome.

**Examples:** The number of dots showing on a rolled die; letters in the alphabet, people in a group, playing cards in a standard deck.

(B) $K$ elements are selected from this collection to construct an outcome. An important distinction is: Do we put the elements back before selecting the next? (yes = “with replacement”; no = “without replacement”).

**Examples:** Roll a die twice, or row 2 dice and observe the dots showing on the face; choose 5 letters, choose people for a committee, deal a poker hand.

(C) The elements are used to construct an outcome, generally
- a **sequence** (a collection with an ordering, with possible duplicates),
- a **set** (an unordered collection without duplicates),
- (also possibly a **bag** or **multiset** (unordered collection with duplicates).

**Example:** Collect 5 selected cards into a poker hand; arrange the 5 letters to make a word.
Finite Combinatorics

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Finite Combinatorics

The important issues to note are (and you probably want to figure them out in this order):

(i) Is the outcome ordered or unordered? Does the outcome have duplicates or not?

**Examples:** If the two numbers showing on the dice are 2 and 5, put them in sequence [2,5] (duplicates allowed); put the 5 letters into a word (a sequence); put the 5 cards into a hand (a set).

Note: in the case of [2,5] you may speak of the "first roll" or the "second roll" but in { 2,5 } you may only make statements about the collection without specifying an order ("at least one of the rolls is 5"). Words are sequences, and hands in card games are sets; otherwise you need to use context or it will be clear from the problem statement.

(ii) Is the selection done with or without replacement?

**Examples:** Selecting 5 cards for a poker hand is done without replacement (you keep the cards and don't put them back in the deck); choosing a committee of 3 from a group of 10 people is without replacement; in many cases, as with balls in a sack, it is part of the problem statement.

(iii) Does the basis collection have duplicates or not?

**Examples:** The collection of letters in "MISSISSIPPI" has duplicates, but in "WORD" there are no duplicates; two red balls in a sack are indistinguishable by color.
Finite Combinatorics

We will organize this along the dimensions of

- ordered vs unordered and
- selection with replacement vs without.

and we will consider the role of duplicates when appropriate.

These problems have names you should be familiar with from CS 131:

<table>
<thead>
<tr>
<th>Ordered Outcome</th>
<th>Selection Without Replacement</th>
<th>Selection With Replacement</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Sequence or String)</td>
<td>Permutations</td>
<td>Enumerations</td>
</tr>
<tr>
<td>Unordered Outcome</td>
<td>Combinations</td>
<td>(We will not study this possibility...)</td>
</tr>
<tr>
<td>(Set or Multiset)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For each of these I will provide a canonical problem to illustrate; I STRONGLY recommend you memorize these problems and the solution formulae, and when you see a new problem, try to translate it into one of the canonical problems.
Finite Combinatorics

Enumerations

The simplest situation is where we are constructing a sequence with replacement, such as where the basis objects are literally replaced, or consist of information such as symbols, which can be copied without eliminating the original.

**Canonical Problem:** You have N letters to choose from; how many words of K letters are there?

**Formula:** $N^K$

**Example:** How many 10-letter words all in lower case? $26^{10}$

A more general version of this involves counting cross-products:

**Generalized Enumerations:** Suppose you have K sets $S_1, S_2, ..., S_k$. What is the size of the cross-product $S_1 \times S_2 \times ... \times S_k$?

**Solution:** $|S_1| \cdot |S_2| \cdot ... \cdot |S_k|$

**Example:** Part numbers for widgets consist of 3 upper-case letters followed by 2 digits. How many possible part numbers are there? $26 \cdot 26 \cdot 26 \cdot 10 \cdot 10 = 1,757,600$
Finite Combinatorics

Permutations

Next in order of difficulty (and not yet very difficult) are permutations, where you are constructing a sequence, but without replacement. This explains what happens when the basis set is some physical collection which can not (like letters) simply be copied from one place to another.

The most basic form of permutation is simply a rearrangement of a sequence into a different order. The number of such permutations of $N$ objects is denoted $P(N,N)$.

**Canonical Problem 1(a):** Suppose you have $N$ students $S_1, S_2, \ldots, S_n$. In how many ways can they ALL be arranged in a sequence in $N$ chairs?

**Formula:** $P(N,N) = N \times (N-1) \times \ldots \times 1 = N!$

**Example:** How many permutations of the word "COMPUTER" are there?

**Answer:** $8! = 40,320$
Finite Combinatorics

K-Permutations

If we do not simply rearrange all N objects, but consider selecting K <= N of them, and arranging these K, we have a “K-Permutation” indicated by P(N,K).

**Canonical Problem 1(b):** Suppose you have N students S₁, S₂, ..., Sₙ. In how many ways can K of them be arranged in a sequence in K chairs?

**Formula:**

\[ P(N, K) = N \times (N - 1) \times \cdots \times (N - K + 1) = \frac{N \times (N - 1) \times \cdots \times (N - K + 1) \times (N - K) \times \cdots \times 1}{(N - K) \times \cdots \times 1} = \frac{N!}{(N - K)!} \]

**Example:** How many passwords of 8 lower-case letters and digits can be made, if you are not allowed to repeat a letter or a digit?

**Answer:** The “not allowed to repeat” means essentially that you are doing this ”without replacement”. So we have P(36,8) = 36! / 28! = 1,220,096,908,800.

**Note:** The usual formula at the extreme right is extremely inefficient. The first formula is the most efficient, if not the shortest to write down!
Finite Combinatorics

Counting With and Without Order

Before we discuss combinations, let us first consider the relationship between ordered sequences and unordered collections (sets or multisets). For example, consider a set

\[ A = \{ S, E, T \} \]

of 3 letters (all distinct). Obviously there is only such set.

There are \(3! = 6\) different sequences of all these letters, but obviously if we can not distinguish the two O’s, then not all these sequences are distinct:

\[
\begin{align*}
S & \ E \ T \\
S & \ T \ E \\
E & \ S \ T \\
E & \ T \ S \\
T & \ S \ E \\
T & \ E \ S
\end{align*}
\]

\textbf{Set} = unordered, no duplicates
\textbf{Multiset} = unordered, maybe duplicates
Finite Combinatorics
The Unordering Principle

If there are $M$ ordered collections of $N$ elements, then there are $M/N!$ unordered collections of the same $M$ elements.

When all elements are distinct, as in our previous example, then obviously, $M/N! = N!/N! = 1$.

The basic idea here is that we are correcting for the overcounting when we assumed that the ordering mattered. Therefore we divide by the number of permutations.

This principle also applies to only a part of the collection:

**Example:** Suppose we have 4 girls and 5 boys, and we want to arrange them in 9 chairs, but we do not care what order the girls are in. How many such arrangements are there?

**Answer:** There 9! permutations, but if we do not care about the order of the (sub)collection of 4 girls, then there are $9!/4! = 15,120$ such sequences.
Finite Combinatorics

Permutations with Repetitions

As another example of the Unordering Principle, let us consider what happens if you want to form a permutation $P(N,N)$, but the $N$ objects are not all distinct. An example may clarify:

**Example:** How many distinct (different looking) permutations of the word “FOO” are there?

If we simply list all $3! = 6$ permutations, we observe that because the ‘O’ is duplicated, and we can not tell the difference between two occurrences of ‘O’ s, there are really only 3 distinct permutations. This should be clear if we distinguish the two occurrences of ’O’ with subscripts:

<table>
<thead>
<tr>
<th>Sequences:</th>
<th>Multiset:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1O_2$</td>
<td>{ O, O }</td>
</tr>
<tr>
<td>$O_1O_2$</td>
<td></td>
</tr>
<tr>
<td>$O_O$</td>
<td></td>
</tr>
<tr>
<td>$O FO$</td>
<td></td>
</tr>
<tr>
<td>$O O F$</td>
<td></td>
</tr>
<tr>
<td>$O O F$</td>
<td></td>
</tr>
</tbody>
</table>

There are 2! sequences, so $6/2! = 6/2 = 3$. 
Finite Combinatorics

Permutations with Repetitions

If you have $N$ (non-distinct) elements, consisting of $m$ (distinct) elements with multiplicities $K_1, K_2, ..., K_m$, that is, $K_1 + K_2 + ... + K_m = N$, then the number of distinct permutations of the $N$ elements is

$$
\frac{N!}{K_1! * K_2! * \cdots * K_m!}
$$

Example: How many distinct (different looking) permutations of the word “MISSISSIPPI” are there?

Solution: There are 11 letters, with multiplicities:

- M: 1
- I: 4
- S: 4
- P: 2

Therefore the answer is

$$\frac{11!}{1! * 4! * 4! * 2!} = \frac{39,916,800}{1 * 24 * 24 * 2} = 34,650$$
Finite Combinatorics

Circular Permutations

A related idea is permutations of elements arranged in a circle. The issue here is that (by the physical arrangement in a circle) we do not care about the exact position of each elements, but only “who is next to whom.” Therefore, we have to correct for the overcounting by dividing by the number of possible rotations around the circle.

Example: There are 6 guests to be seated at a circular table. How many arrangements of the guests are there?

Hint: The idea here is that if everyone moved to the left one seat, the arrangement would be the same; it only matters who is sitting next to whom. So we must factor out the rotations. For N guests, there are N rotations of every permutation.

Solution: There are 6! permutations of the guests, but for any permutation, there are 6 others in which the same guests sit next to the same people, just in different rotations.

Formula: There are \( \frac{N!}{N} = (N - 1)! \) circular permutations of N distinct objects.