Binomial Distribution

Recall that the Binomial Distribution is omnipresent in statistics, probability, and even in physical phenomena:

https://www.mathsisfun.com/data/quincunx.html
But it is hard to manipulate when the numbers get big or when the probabilities \( p \) or \( (1-p) \) get too close to 0:

\[
\begin{align*}
\text{Binomial Distribution} & \\
\text{But it is hard to manipulate when the numbers get big or when the probabilities } & \text{get too close to 0:}
\end{align*}
\]

In [611]: b(1000)

File ...

How to approximate the binomial? We have seen that under certain conditions, the Poisson can be used to approximate the binomial:

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Binomial Distribution

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![Binomial Distribution Chart](chart1.png)

Binomial Distribution

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![Binomial Distribution Chart](chart2.png)
Binomial Distribution

How to approximate the binomial? When we observe the characteristic shape of the Binomial Distribution as \( N \) approaches infinity, we see something interesting:

![Binomial Distribution Graph]

Normal Distribution

The Binomial Distribution in the limit looks like the continuous Normal Distribution, which is a particular kind of Gaussian Exponential:

![Normal Distribution Graph]
Gaussian Exponential Function

We will use exponential functions in the next lecture, so let’s think about what that means.

Here is a graph of $e^x$, where $e = 2.71828$ (Euler’s Constant):

![Graph of $e^x$](image1)

Gaussian Exponential Function

Here is a graph of $e^{-x}$, which flips the function around the Y axis (this is a simple example of the Exponential Distribution, which we will examine in detail next time):

![Graph of $e^{-x}$](image2)
Gaussian Exponential Function

But we want to make it look like the Binomial Distribution, so it has to be symmetric around the Y axis, so we'll reflect it around the y axis with the absolute value:

Graph of $e^{-|x|}$:

But we want to have a smooth curve, so we'll use the square instead of the absolute value:

Graph of $e^{-x^2}$
Gaussian Exponential Function

By adding parameters to adjust the height and the width, we have the Gaussian Exponential Function:

\[ f(x) = ae^{-\frac{(x-b)^2}{2c^2}} \]

This function is widely used to describe phenomena (light, electric charge, gravity, quantum probabilities) that decrease in effect with distance, to create filters in signal processing (e.g., audio programming, graphics), etc., in addition to their wide use in statistics and probability.

Normal Distribution

By using parameters to fit the requirements of probability theory (e.g., that the area under the curve is 1.0), we have the formula for the Normal Distribution, which can be used to approximate the Binomial Distribution and which models a wide variety of random phenomena:

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ \sigma = \sqrt{\text{Variance}} \]

so……

\[ \sigma^2 = \text{Variance} \]
By using parameters to fit the requirements of probability theory (e.g., that the area under the curve is 1.0), we have the formula for the **Normal Distribution**, which can be used to approximate the Binomial Distribution and which models a wide variety of random phenomena:

The 4 most important things to remember are that:

1. the Normal Distribution $N(\mu, \sigma^2)$ is defined directly by the mean and variance,
2. it is symmetric around the mean;
3. we calculate probabilities by areas, NOT points; and
4. we can determine probabilities knowing only the distance from the mean in terms of the standard deviation.

For the normal distribution, the values less than one standard deviation away from the mean account for 68.27% of the set; while two standard deviations from the mean account for 95.45%; and three standard deviations account for 99.73%. 

$\text{Normal Distribution } N(\mu, \sigma^2)$ for various values of $\sigma$: 

![Normal Distribution Graph](image)
Recall that the only way we can analyze probabilities in the continuous case is with the CDF:

\[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \]

\[ F(x) = \int_{-\infty}^{x} f(y) \, dy \]

- \[ P(X < b) = F(b) \]
- \[ P(X > a) = 1.0 - F(a) \]
- \[ P(a < X < b) = F(b) - F(a) \]
Since there are a potentially infinite number of Normal Distributions, sometimes we calculate using a normalized version, the **Standard Normal Distribution** with mean 0 and standard deviation (and variance) 1:

\[ N(0, 1) \]

Any random variable \( X \) which has a normal distribution \( N(\mu, \sigma^2) \) can be converted into a standardized random variable \( Z \) with distribution \( N(0,1) \) by the following formula:

\[ Z = \frac{X - \mu}{\sigma} \]

This is usually helpful in **HAND** calculations, since \( \mu \) and \( \sigma \) have been factored out......

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**Normal Distribution**

If you were doing these calculations in 1900, or haven’t heard of a computer, here is how you would calculate the probability of a normally-distributed random variable:
Normal Distribution

If you were doing these calculations in 1900, or haven’t heard of a computer, here is how you would calculate the probability of a normally-distributed random variable:

But since we know how to use computers, we’ll do it a better way.....

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<th>0.0</th>
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<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
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<td>0.9978</td>
<td>0.9979</td>
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</tr>
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</table>

Note: Sometimes \( \Phi(x) \) or \( \Phi(x) \) is used instead of \( F(x) \) for the CDF.

```python
def Phi(mu, var, x):
    return (1 + math.erf((x-mu)/(var**0.5 * 2.0**0.5))) / 2

def normal_range(mu, var, low, high):
    return Phi(mu, var, high) - Phi(mu, var, low)
```
Normal Distribution

Thus, calculating the probability of being between two ranges in the normal distribution is simply plugging in values to the appropriate function:

Ex 1: Suppose weights are normally distributed with mean 137.5 and standard deviation 15.7. What is the probability that a random person weighs less than 100 lbs?

```
In [918]: normalRange(125,25,128,130)
Out[918]: 0.6826884921170861
In [919]: normalRange(125,25,115,135)
Out[919]: 0.9544997361836414
In [920]: normalRange(125,25,110,140)
Out[920]: 0.9973002809367398
```

Ex. Suppose weights are normally distributed with mean 137.5 and standard deviation 15.7. What is the probability that someone weighs 150 lbs or more?

```
In [921]: 1 - Phi(137.5, 15.7*15.7, 150)
Out[921]: 0.2129641914683208
```