You must complete 5 of the 6 problems on this exam for full credit. Each problem is of equal weight. Please leave blank, or draw an X through, or write “Do Not Grade,” on the problem you are eliminating; I will grade the first 5 I get to if I can not figure out your intention—no exceptions! If answers are on the back of the page please tell me so.

**Circle final answers.** No calculators allowed, and you may leave complicated formulae uncomputed, but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself.

In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you can not completely solve the problem, show me as much as you know and I will attempt to give you partial credit.

**Problem One**

Suppose a discrete random variable $X$ is distributed according to the uniform discrete distribution in the range $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$, that is,

$$f_X(k) = \begin{cases} 1/9 & \text{if } -4 \leq k \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

Furthermore, let the random variable $Z = X^2$.

(a) Draw the CDF of $X$.

(b) Give the range $R_Z$ and probability function $f_Z$ for $Z$. 

$$R_Z = \{3, 9, 16\}$$

$$f_Z = \begin{cases} \frac{1}{9} & \text{if } 3 \\ \frac{1}{9} & \text{if } 9 \\ \frac{1}{9} & \text{if } 16 \end{cases}$$
(c) What is $E[Z]$? Show all work.

\[ E[Z] = \frac{2}{9} + \frac{8}{9} + \frac{16}{9} + \frac{32}{9} = \frac{60}{9} = \frac{20}{3} \]

(d) What is $\text{Var}(X)$? Show all work.

\[ \text{Var}(X) = E(X^2) - E(X)^2 \]
\[ = E(Z) - \left( \frac{20}{3} \right)^2 \]
\[ = \frac{20}{3} - \left( \frac{20}{3} \right)^2 \]
\[ = \frac{20}{3} - \frac{400}{9} = \frac{60}{9} - \frac{400}{9} = \frac{-340}{9} \]
**Problem Two**  Let X be a random variable defined as follows. Toss a die: if the number of dots showing is odd, let \( Y = 1 \), otherwise let \( Y = 3 \).

(A) What distribution does X follow? Give the range \( \mathcal{R}_X \) and probability distribution \( f_X \) for X.

This is Bernoulli(0.5):

\[
\mathcal{R}_X = \{1, 3\}
\]
\[
f_X = \{\frac{1}{2}, \frac{1}{2}\}
\]

(B) Calculate explicitly (not just by quoting a formula) the expected value \( E(X) \) and show every step. You may leave the result as a fraction.

\[
E(X) = \frac{1}{2} + \frac{3}{2} = 2
\]

(C) Calculate the variance \( \text{Var}(X) \) and standard deviation \( \sigma_X \) of X. Again, do not just quote a formula, but show explicitly all calculations. You may leave the result as a fraction.

\[
\text{Var}(X) = E(X^2) - E(X)^2
\]
\[= \left(\frac{1}{2} \cdot 1^2 \right) + \left(\frac{1}{2} \cdot 3^2 \right) - 2^2
\]
\[= \frac{1}{2} + \frac{9}{2} - 4
\]
\[= \frac{5 - 8}{2} = 1
\]

\[
\sigma_X = 1
\]
**Problem Three**

A box contains 10 coins where 5 coins have a head on each side, 3 coins have a tail on each side and 2 are fair coins (head and tail with 50% chance of each when tossed).

A tree diagram is the best way to start this problem.

(A) Suppose a coin is chosen at random and tossed. Find the probability that a head appears.

(B) Suppose we repeat the experiment in (A) 10 times. What is the probability that at least 7 heads appear? (You may just give the formula.)

(C) Suppose a coin is selected at random and tossed. If a head appears, find the probability that the coin was fair, i.e., one with a head on one side and tail on the other.
Problem Four

Wayne and Lenka are playing a game in which each has a fair coin, and they flip the coins at the same time, and keep doing so until they have the same face showing (both heads or both tails). A round is one simultaneous flip. For example, they might have the following:

L: H T H
W: T H H  (three rounds = three flips)

or they might have:

L: H
W: H  (one round = one flip)

(A) If \( X = \) the number of rounds the game lasts, what is the distribution of \( X \)? Be absolutely precise.

\[
X \sim \text{Geo}(P = \frac{1}{2})
\]

\[
P(x\leq3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}
\]

(B) What is the expected number of rounds when they play this game?

\[
E(X) = \frac{1}{P} = \frac{1}{\frac{1}{2}} = 2
\]

(C) What is the probability that the game lasts more than 1, but less than 6 rounds?

\[
P(1 < X < 6) = P(X > 1) - P(X > 5)
\]

\[
P(1 < X < 6) = (1 - \left(\frac{1}{2}\right)^1) - (1 - \left(\frac{1}{2}\right)^5)
\]

\[
P(1 < X < 6) = \frac{1}{2} - \frac{1}{32} = \frac{15}{32}
\]

(D) What is the probability that if the game lasts exactly 7 rounds, that Wayne has landed heads more times than Lenka? You may calculate it or guess, but if you guess you must give a reason for your guess.

\[
\text{Guess:} \frac{1}{2} \quad \text{Reason:}
\]

\[
\text{Game is symmetric (could exchange heads & tails without changing probs)}
\]

\[
\text{So:} \frac{1}{2}
\]
Problem Five

Suppose X ~ Uniform(0,10), i.e., it produces a random real number between 0 and 10 according to the following graph of the Probability Density Function f_X(a) (the y axis has been left off on purpose):

(a) Give of f_X(a) as a mathematical formula.

\[ f_X(a) = \begin{cases} \frac{1}{10} & \text{for } 0 \leq a \leq 10 \\ 0 & \text{otherwise} \end{cases} \]

(b) Calculate the Cumulative Distribution Function F_X(a) using integrals and draw the graph.

\[ F_X(a) = \int_{0}^{a} \frac{1}{10} \, dx = \frac{1}{10}x \bigg|_{0}^{a} = \begin{cases} 1 & \text{for } a > 10 \\ \frac{a}{10} & \text{for } 0 \leq a \leq 10 \\ 0 & \text{for } a < 0 \end{cases} \]

(c) Calculate E(X) using integrals (you can probably guess what it is, but I want you to derive it using integrals).

\[
E(X) = \int_{0}^{10} x \cdot \frac{1}{10} \, dx = \frac{1}{20}x^2 \bigg|_{0}^{10} = \frac{100}{20} = 5.0
\]
Problem Six

There are 9 students in a class, and they need to be divided into 3 teams to play a contest.

You may leave these answers as formulae.

(a) Suppose that they need to be divided into 3 equal-sized teams named “Reds”, “Blues”, and “Blacks.” How many ways can this be done?

\[
\binom{9}{3} \binom{6}{3} \binom{3}{3} = 1680
\]

(b) Now suppose they need to be divided into one team of 5, and two teams of 2 each. How many ways can this be done?

\[
\binom{9}{5} \binom{4}{2} \binom{2}{2} = \frac{720 \times 6 \times 1}{2} = 360
\]

(c) Now suppose you need to divide the class into 3 equal-sized teams, and for each team you need to select a captain. How many ways can this be done? (Realize that once you select the members of each team, there are multiple ways to select the captains of each team.)

\[
\binom{9}{3} \binom{6}{3} \binom{3}{3} \binom{3}{3} \binom{3}{3} = 3!
\]

Since same size and no names, eliminate all permutations or 3!!