CS 237—Midterm Exam Solutions
Spring 2014

You must complete 4 of the 6 problems on this exam for full credit. Each problem is worth 25 points. Please leave blank, or draw an X through, or write “Do Not Grade,” on the two problems you are eliminating; I will grade the first 4 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers.** No calculators allowed, and you may leave *complicated* arithmetic expressions uncomputed, but please do multiply 1/2 * 1/2 to get 1/4 if the occasion presents itself. In composing your answers, remember that your goal is to show me you understand the techniques presented in the course; if you solve a problem using an inappropriate technique, you may not receive full credit. *In particular, if you simply enumerate all cases and count, avoiding all math other than arithmetic, you will receive a grade appropriate to the level of understanding you have demonstrated.*

**Problem One.** Suppose that the population of Rich Hall is surveyed about their preferences for breakfast, and 47% eat eggs, 33.4% eat pancakes, 34.6% eat cereal, 11.9% eat eggs and pancakes, 15.1% eat eggs and cereal, 10.4% eat pancakes and cereal, and 4.8% eat all three. If I select a student at random from Rich Hall, write the *formula* (not a diagram, though you may create one for inspiration) for the probability that he/she eats neither eggs, pancakes, nor cereal for breakfast, where

- **E** = event that the student eats eggs,
- **P** = event that the student eats pancakes, and
- **C** = event that the student eats cereal.

**Solution:** We employ the Inclusion-Exclusion Principle to find 1.0 - P(E ∪ P ∪ C), where:

\[
P(E ∪ P ∪ C) = P(E) + P(P) + P(C) - P(E ∩ P) - P(E ∩ C) - P(P ∩ C) + P(E ∩ P ∩ C)
\]

\[
= 0.47 + 0.334 + 0.346 - 0.119 - 0.151 - 0.104 + 0.048 = 0.824
\]

Thus, 1.0 – 0.824 = 0.176 or 17.6%.
**Problem Two** A family has 5 children. Find the probability that there are fewer boys than girls. (Assume the probability of any particular child being a boy is 0.5.)

**Solution:** This is B(5, 0.5) and we are looking for
\[
P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2)
\]
\[
= \binom{5}{0}0.5^00.5^5 + \binom{5}{1}0.5^10.5^4 + \binom{5}{2}0.5^20.5^3
\]
\[
= 0.5.
\]
This is intuitively obvious, since the situation is symmetric (i.e., the probability would be the same if we said “fewer girls than boys”).

**Problem Three.** In a small town, 11 of the 25 schoolteachers are against the legalization of marihuana, eight are for the legalization of marihuana, and the rest are indifferent. A random sample of 5 schoolteachers is selected to be interviewed for the local paper on the issue. What is the probability that all 5 share the same opinion?

**Solution:** We simply calculate the probabilities for each of the three possible shared opinions and add:
\[
\frac{C(11,5)}{C(25,5)} + \frac{C(8,5)}{C(25,5)} + \frac{C(6,5)}{C(25,5)} = 0.0099
\]

**Problem Four.** From families with three children, a family is selected at random and found to have a boy. What is the probability that the boy has an older brother and a younger sister?

**Solution:** The sample space is \{BBB, BBG, BGB, BGG, GBB, GBG, GGB, GGG\}, where we assume they are in order of age from left to right. The probability we seek is \(P(A|B)\) where \(A = \text{the boy has an older brother and a younger sister}\) and \(B = \text{the family selected has a boy}\), where \(P(A) = 1/8\) and \(P(B) = 7/8\). Since \(AB = A\), we just calculate \(P(A)/P(B) = 1/7\).

**Problem Five.** From an ordinary deck of 52 cards we draw cards at random, with replacement, and successively until a face card (Jack, Queen, or King) is drawn. What is the probability that at least 5 draws are needed?

**Solution:** This is the Geometric Distribution \(G(3/13)\) and we seek the tail probability \(P(X > 4)\). The simplest calculation is to find the probability of 4 non-face cards or \((10/13)^4 = 0.3501\). Or, you could use Exclusion and count the probabilities of the cases to exclude:
\[
1.0 - ( P(X=1) + P(X=2) + P(X=3) + P(X=4)) \\
= 1.0 - ( \frac{3}{13} + \frac{10}{13}(\frac{3}{13}) + (\frac{10}{13})^2(\frac{3}{13}) + (\frac{10}{13})^3(\frac{3}{13}) ) \\
= 1.0 - (0.2308 + 0.1775 + 0.1365 + 0.1050) \\
= 0.3501
\]
**Problem Six.** A fair die is rolled twice. Let A denote the event that the sum of the outcomes is odd, and B denote the event that on the first toss it shows a 2. Are A and B independent? Why or why not? (You must answer this mathematically.)

**Solution:** The two events are independent. For A, there are 2 ways to get a 3, 2 ways to get 11, 4 ways to get 5, 4 ways to get 9, and 6 ways to get 7, so \( P(A) = \frac{18}{36} = \frac{1}{2} \).

\( P(B) = \frac{1}{6} \).

We can establish independence in several (equivalent) ways:

1. There are 3 outcomes where the first roll is a 2, and the total is odd, so
   \[
   P(AB) = \frac{3}{26} = \frac{1}{12} = P(A) \times P(B)
   \]

2. The conditional probability \( P(A|B) = \frac{1}{2} \) (since half the second roll’s outcomes are odd), and thus
   \[
   P(A|B) = P(A)
   \]

3. Finally, \( P(B|A) = \frac{3}{18} = \frac{1}{6} \) (since, of the 18 odd outcomes, 3 are due to a 2 on the first roll) and so
   \[
   P(B|A) = P(B)
   \]