Music is fashioned wholly in the likeness of numbers. Whatever is delightful in song is brought about by number. Sounds pass quickly away, but numbers, which are obscured by the corporeal element in sounds and movements, remain.
—“Scholia Enchiriadis”

It has been written that the shortest and best way between two truths of the real domain often passes through the imaginary one.
—Jacques Hadamard

Mathematicians aren’t above imagining new kinds of numbers when circumstances warrant. The natural numbers—whole numbers greater than zero—are probably as old as civilization. But when the natural numbers failed to solve equations such as $c = a - b$ for all possible natural numbers $a$ and $b$, mathematicians invented negative numbers. The result was the birth of the integers. Rational numbers were developed when integers failed to solve equations such as $c = a + b$ for all possible integers $a$ and $b$. When a careful look at irrational numbers such as $\pi$ showed the limitations of rational numbers, mathematicians invented real numbers. Yet there are straightforward mathematical situations that can’t be handled by real numbers either.

### 2.1 Why Imaginary Numbers?

Table 2.1 shows a sequence of simple equations requiring increasingly advanced number systems for their solution, starting with natural numbers and progressing through solutions requiring real numbers. But what about that last equation? How can it be solved?

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution</th>
<th>Requires</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x = 4$</td>
<td>$x = 2$</td>
<td>Natural integers</td>
</tr>
<tr>
<td>$2x + 4 = 0$</td>
<td>$x = -2$</td>
<td>Signed integers</td>
</tr>
<tr>
<td>$4x = 2$</td>
<td>$x = 1/2$</td>
<td>Rational numbers</td>
</tr>
<tr>
<td>$x^2 = 2$</td>
<td>$x = \sqrt{2}, x = -\sqrt{2}$</td>
<td>Real numbers</td>
</tr>
<tr>
<td>$x^2 + 2 = 0$</td>
<td>$x = ?$</td>
<td>?</td>
</tr>
</tbody>
</table>
First, rearrange it a bit:

\[ x^2 = -2. \]  
\[ (2.1) \]

Solving for \( x \) means taking the square root of both sides. This would require a number on the right-hand side that, when squared, results in a negative number. But squaring is multiplying a number by itself, and the result is always positive. So there can be no solution to (2.1) under the standard rules of mathematics. Although (2.1) is simple to write, its solution poses a contradiction to familiar mathematical understanding.

What if we attempted to “quarantine” the minus sign, to factor it away from the 2? For example, we can rewrite equation (2.1) as \( x^2 = 2(-1) \). In algebra we frequently let a variable or an expression stand for an unknown quantity. Let’s put the unknown and troublesome aspect of (2.1) into this expression: \( i^2 = (-1) \). Using this definition, we can rewrite (2.1) as \( x^2 = 2 \cdot i^2 \). Solving for \( x \), we obtain \( x = \sqrt{2} \cdot i \).

We managed to banish the minus sign by embedding it in \( i \), but we’re hardly any better off because we don’t know how to interpret \( i \), which is still unknown. All we know is that it would have to be a number that, when multiplied by itself, equals \(-1\). But we also know that there is no such number under the rules of mathematics we currently understand them.

We have a choice: stick with the rules we have (which we’d very much like to do) or fiddle around with the rules (which might lead to chaos, so we’d rather not). But having observed that the current rules do not cover all outcomes, we can’t just ignore this problem.

So, consider inventing a new kind of number that, when multiplied by itself, produces a negative result. We’d have to modify the rules of mathematics carefully to allow such a number without falsifying anything we already know to be true.

To pick up where we left off, let

\[ i^2 = -1. \]  
\[ (2.2) \]

Now let’s assert that the square root of \( i^2 \) is

\[ i = \sqrt{-1}. \]  
\[ (2.3) \]

Without a doubt, equations (2.3) and (2.4) stand conventional mathematics on its ear. But these turn out to be just what we need. Start with equation (2.2): \( x = \sqrt{2} \cdot i \). Square it to prove that it leads back to equation (2.1).

\[ x^2 = (\sqrt{2} \cdot i)^2 \quad \text{Square both sides.} \]
\[ = (\sqrt{2})^2 \cdot i^2 \quad \text{Square the terms separately because } (ab)^2 = a^2 \cdot b^2. \]
\[ = (\sqrt{2})^2(-1) \quad \text{Substitute } -1 \text{ for } i^2 \text{ by equation } (2.3). \]
\[ = 2 \cdot (-1) \]
\[ = -2. \]
Using (2.3), the Imaginary Rule, we have found a solution to \( x^2 = -2 \). But what sort of number is created in equation (2.4) to facilitate this solution? It is certainly not a real number because no real number when multiplied by itself produces a negative result. Mathematicians have named \( \sqrt{-1} \) the *imaginary number* (although it is worth pointing out that in fact all numbers are imaginary insofar as they are all free creations of the human mind). The usefulness of the imaginary number is actually quite real; its use leads to some particularly beautiful insights about music and sound.

### 2.2 Operating with Imaginary Numbers

Let’s take a moment to summarize. In order to solve all the equations in table 2.1, we had to create a new kind of number—the imaginary number—which produces a negative result when squared. So far, there’s one number in this entire class of numbers, \( i = \sqrt{-1} \).

This is unsettling because we have to relate this new kind of number to all other kinds. Clearly, if we have some numbers that produce only positive results when squared and others that produce negative results, we must figure out how to tell them apart and keep them distinct. Some things we already know about imaginary numbers can help.

For instance, whenever the imaginary number is multiplied by a real number, the result is also an imaginary number. For example, if \( x = \sqrt{2} \cdot i \), then \( x \) is also an imaginary number. We can prove this by squaring it: if the result is negative, \( x \) is imaginary, and if the result is positive, \( x \) must be a real number. This was already demonstrated, but here it is again:

\[
x^2 = (\sqrt{2}i)^2
= 2(-1)
= -2.
\]

So, we know several facts about multiplying imaginary numbers.

**The product of an imaginary number and a real number is an imaginary number.**

**Squaring an imaginary number turns it back into a real number.**

**If the square of a number is negative, then it is imaginary.**

But can we combine imaginary numbers with other mathematical operations, like addition? We can’t add real numbers and imaginary numbers any more than we can directly add apples and oranges because their differing characteristics require that we treat them distinctly. However, we could add apples and oranges together if we kept them distinct. Imagine the following conversation:

“Hi. How many apples do you have in your bag?”

“Five.”

“And how many oranges do you have?”
“Four.”
“Here, take these three apples and these three oranges; now how many do you have?”
“I have eight apples and seven oranges.”

This conversation can be notated using $a$ for apples and $o$ for oranges: $(5a + 4o) + (3a + 3o) = 8a + 7o$. Everything is fine so long as we preserve the distinction between apples and oranges and operate on them separately. We could even formalize this rule by defining a new class of object named “fruit” and declaring that “fruit consists of a certain number of apples plus a certain number of oranges.” For example, we could have the following quantities of fruit: $(5a + 4o)$ or $(3a + 3o)$ or, if we had none, $(0a + 0o)$ or, if I had no apples and owed you an orange, $(0a - 1o)$.

We can combine real numbers and imaginary numbers the same way. Just as we defined the term fruit to mean the combination of apples and oranges, mathematicians have adopted the term complex numbers to mean the combination of a regular number and an imaginary number. I think this name is unfortunate because it suggests these numbers are complicated. In fact, complex numbers are no more complicated than regular numbers.

### 2.3 Complex Numbers

In the preceding example, I represented a sum of apples and oranges using a notation that allowed combining them but keeping them distinct. We can take a similar approach to constructing complex numbers.

A complex number is the sum of a real and an imaginary number. The imaginary part of the sum is distinguished by $i$.

Table 2.2 shows some examples of complex numbers. The third column shows how these numbers are sometimes abbreviated in practice.

#### 2.3.1 Operations on Complex Numbers

We need a way to isolate parts of complex numbers so we can take them apart. We do so with two functions, Re and Im. If $z = a + bi$, then

- $a = \text{Re}(z)$. Evaluates to the real part of $z$.
- $b = \text{Im}(z)$. Evaluates to the imaginary part of $z$ as a real number.

<table>
<thead>
<tr>
<th>Complex Number</th>
<th>Description</th>
<th>Short Equivalent</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 + 0i$</td>
<td>Zero real part and zero imaginary part</td>
<td>$0$</td>
</tr>
<tr>
<td>$0 + bi$</td>
<td>Zero real part and imaginary part with value $b$</td>
<td>$bi$</td>
</tr>
<tr>
<td>$0 + li$</td>
<td>Zero real part and imaginary part of $1$</td>
<td>$i$</td>
</tr>
<tr>
<td>$a + (-b)i$</td>
<td>Real part $a$ and imaginary part $-b$</td>
<td>$a - bi$</td>
</tr>
</tbody>
</table>

This rather section 2.3.

Complex numbers are of the form $u + vi$ for multiply product, an

- $u \cdot v = (a \cdot c) + (b \cdot d)i$ for multiplication.

The complex number $-u = -(a, b)$ is convenient.
Musical Signals

For example,
\[ \text{Re}\{a + (-b)i\} = a, \]
\[ \text{Im}\{a + (-b)i\} = -b. \]

Notice that the result returned by \( \text{Im}\{ \} \) is the imaginary part converted back into a real number. That is, \( \text{Im}\{a + bi\} = b \), not \( bi \).

Remembering always to keep the real and imaginary parts of a complex number distinct, we can make up rules for how complex numbers behave under standard mathematical operations. First, let \( u = a + bi \) and \( v = c + di \), where \( a \) and \( c \) are the real parts and \( bi \) and \( di \) are the imaginary parts of \( u \) and \( v \), respectively. We now can define the following operations.

**Complex Equality** For two complex numbers to be equal, their real parts must match *and* their imaginary parts must match. That is, if \( u = v \), then \( a = c \), and \( b = d \).

**Complex Addition** If we add two complex numbers, we must add the real parts and the imaginary parts separately:

\[
\begin{align*}
  u + v &= (a + bi) + (c + di) \\
        &= (a + c) + (bi + di) \\
        &= (a + c) + (b + d)i. \\
\end{align*}
\]

*Complex Addition* (2.6)

**Complex Multiplication** If we multiply two complex numbers, we follow the usual procedures for multiplication, remembering that a real number times an imaginary number yields an imaginary product, and that an imaginary number squared yields a negative product:

\[
\begin{align*}
  u \cdot v &= (a + bi)(c + di) \\
           &= ac + bdi^2 + adi + bci \\
           &= ac - bd + bci + adi \\
           &= (ac - bd) + (bc + ad)i. \\
\end{align*}
\]

*Complex Multiplication* (2.7)

This rather complicated arithmetic result will become a lot clearer with a graphical technique (see section 2.3.4).

**Complex Negation** Negating inverts the sign of both the real and the imaginary parts of the complex number \( u \):

\[ -u = -(a + bi) = -a - bi. \]

*Complex Negation* (2.8)

**Complex Conjugation** Since complex numbers provide two sign values to operate on, it would be convenient to be able to change the sign of the real part or the sign of the imaginary part independently.

*The conjugate of a complex number is the negation of its imaginary part.*
I indicate the conjugation operation by putting a bar over the quantity to be conjugated:

\[ \bar{z} = \overline{a+bi} = a - bi, \]

*Complex Conjugate* (2.9)

which reads, "The complex conjugate of \( a + bi \) is \( a - bi \)."

When we multiply a complex number \( (a - bi) \) and its conjugate, the imaginary component drops out, and the result is real:

\[
(a - bi)(a + bi) = a^2 + abi - abi - b^2i^2
\]

\[= a^2 + b^2.\]

*An complex number multiplied by its complex conjugate is a real value.*

**Complex Division**  Dividing a complex number by a real number is easy; divide the imaginary and real parts separately. For example, \( (6 + 8i)/2 = 3 + 4i \). But what if the denominator is not real?

Sometimes it's easier to work around a problem than face it head-on. What if we could make the imaginary component drop out from the denominator? Then the problem would revert to trivial division by a real value. We know we can convert a complex number into a real number by multiplying it by its conjugate. But whatever we do to the denominator we must also do to the numerator to keep the balance. For example, find the dividend of

\[
\frac{2 + 2i}{3 - 2i}.
\]

Multiplying the numerator and denominator by the complex conjugate of the denominator will make the denominator real and allow us to divide the complex numerator by a real denominator:

\[
\frac{(2 + 2i)(3 + 2i)}{(3 - 2i)(3 + 2i)} = \frac{6 + 10i + 4i^2}{9 - 4i^2} = \frac{2 + 10i}{13} = \frac{2}{13} + \frac{10}{13}i.
\]

Consider \( a + bi \) divided by \( c + di \):

\[
\frac{(a + bi)(c - di)}{(c + di)(c - di)} = \frac{(ac + bd) + (bc - ad)i}{c^2 + d^2} + \frac{(bc - ad)i}{c^2 + d^2}.
\]

*Complex Division* (2.10)

Fortunately, we won’t be needing this rather complicated equation because there is a much simpler method for complex division (see section 2.3.6). Meanwhile, the following rule may be helpful.

To divide complex numbers, multiply the numerator and denominator by the conjugate of the denominator, then reduce.

**2.3.2 Graphical Representation of Complex Numbers**

If carrying around two numbers in order to represent one complex number seems difficult, remember we do this with rational numbers, too. Like rational numbers, complex numbers combine two numbers to create a new kind of quantity.
Number pairs also combine to create a new quantity in plane geometry: a point in the Cartesian plane is defined as a pair of numbers, one for the $x$-axis, and one for the $y$-axis. For example (figure 2.1), a point $p$ on the plane can be defined by a combination of horizontal and vertical values $(a, b)$.

If we add Cartesian point $p$ to another point $q$, we must add the $x$ and $y$ values separately. That is, if $p = (a, b)$, and $q = (c, d)$, then

$$p + q = [(a + c), (b + d)].$$

This is like operating on the two halves of a complex number separately. In fact, this suggests a way to interpret complex numbers graphically.

If we associate real numbers with the $x$-axis and imaginary numbers with the $y$-axis, then we could think of a complex number as forming a point on the complex plane. For example, we could associate the complex number $z = a + bi$ with the point $z = (a, b)$ in the Cartesian plane (figure 2.2). This would allow us to apply geometry to understand complex numbers.

Let’s look at some examples. Numbers like 1 and 3.14 are pure real numbers (complex numbers with a 0 imaginary part) lying on the real axis, whereas numbers like $1i$ and $3.14i$ are pure imaginary (with a 0 real part) lying on the imaginary axis. All other numbers $z = a + bi$, such that $a \neq 0$ and $b \neq 0$, are complex numbers in the complex plane. Note that since $i = 1i$, $i$ is represented graphically as 1 on the imaginary axis (figure 2.3). The figure shows the position of some constants on the complex plane and the complex number $z$, its negation $-z$, its conjugate $\bar{z}$, and its negated conjugate $-\bar{z}$. 
2.3.3 Trigonometric Representation of Complex Numbers

Complex numbers really begin to pay off when we view them through trigonometry. They provide a compact, powerful representation for sinusoids that we will come to depend upon. Then the complicated algebraic rules of complex math become unnecessary.

Figure 2.3 shows a triangle in the complex plane. The location of the complex point \( z \) can be found in two ways. First, using geometry, if we define the lengths of the sides of the triangle as \( a \) and \( b \), then the point is \( z = a + bi \).

Equivalently, we could find \( z \) by determining the magnitude (length) of the hypotenuse \( r \) and its angle \( \theta \) above the horizontal plane. If \( r \neq 0 \), and \( \theta \) is the angle of the hypotenuse relative to the positive real axis, then by trigonometry,

\[
\begin{align*}
  a &= r \cdot \cos \theta, \\
  bi &= r \cdot i \sin \theta.
\end{align*}
\]

(See appendix section A.2 for an introduction to trigonometry.)

Substitute these trigonometric definitions for \( a \) and \( bi \) back into the geometric definition for \( z \):

\[
z = a + bi = (r \cdot \cos \theta) + (r \cdot i \sin \theta).
\]

Factoring out the common term \( r \),

\[
z = r(\cos \theta + i \sin \theta). \quad \text{Trigonometric Form of a Complex Number (2.11)}
\]
Equation (2.11) provides a way to find a complex number just by knowing its distance \( r \) from the origin of the complex plane and its angle \( \theta \). More important, (2.11) suggests that we can treat complex numbers as vectors. A vector is simply the combination of a magnitude and a direction. Equation (2.11) identifies a complex number as a magnitude \( r \) and a direction \( \theta \). Furthermore, using (2.11), we can view any complex number equally as the sum of two orthogonal vectors (lying at a 90° angle from one another), the first on the real axis with a magnitude of \( a \) and the second on the imaginary axis with a magnitude of \( b \).

The larger significance of equation (2.11) is that, by relating trigonometric functions to the construction of complex numbers, we now have a bridge between these two realms.

The Angle and the Magnitude In figure 2.4 the variable \( \theta \) is called the angle of \( z \). The variable \( r \), showing the distance from the point \( z \) to the origin, is called the absolute value or the magnitude of \( z \). The latter definition does not conflict with the absolute value of a real number, which is similarly the distance from a point to the origin; we’re just expanding the definition to cover points other than those on the real line. The absolute value of a complex number \( z \) is written \( r = |z| \), just as with integers and real numbers. For example, \( |-3| = 3 \), and \( |3i| = 3 \). But what about complex numbers that do not lie on the real or imaginary axis?

Suppose we have determined the location of a point in the complex plane by its magnitude \( r \) and angle \( \theta \), but now we wish to rediscover its Cartesian coordinates \( a \) and \( b \). Trigonometry again comes to the rescue because this is the same as finding the length of the sides of a right triangle from the length of its hypotenuse and its angle. The length of the side along the real axis is \( a = r \cos \theta \), and the length of the side along the imaginary axis is \( b = r \sin \theta \). So now we can convert back and forth between the complex and Cartesian coordinate systems.

Pythagoras Revisited Another useful relation between the complex number \( z = a + bi \) and the magnitude of its hypotenuse \( r \) involves \( z \) and its conjugate \( \bar{z} \). The relation is \( r^2 = z \bar{z} \). Here’s how to see it. Define a right triangle anchored at the origin of the Cartesian plane with sides \( a \) and \( b \), and hypotenuse \( r \) (figure 2.5). Then by the Pythagorean theorem, we can write
\[
r^2 = a^2 + b^2.
\]

\[
r = \sqrt{2 \bar{z}}
\]

\[
r^2 = a^2 + b^2 = (a+bi)(a-bi) = z \cdot \bar{z}
\]
Now add the term $abi - abi$ to the middle of the right-hand side. Clearly we can do this because adding and then subtracting the same value from an equation does not alter the equality. So we have

$$r^2 = a^2 + abi - abi + b^2.$$  \hfill (2.12)

Focus on the $b^2$ term for a moment. Because of the nature of $i$,

$$b^2 = -b^2 + i^2 = bi(-bi).$$  \hfill (2.13)

We can do this because by equation (2.3), $i^2 = -1$. Repeating equation (2.12) but substituting the definition for $b^2$ from (2.13), we get

$$r^2 = a^2 + abi - abi + bi(-bi).$$

Factoring,

$$r^2 = (a + bi)(a - bi)$$

$$= z\bar{z},$$

by the definition of the complex conjugate.\footnote{The complex conjugate of a complex number $a + bi$ is $a - bi$.} Now if $r^2 = z\bar{z}$, then it must be that the hypotenuse $r = \sqrt{z\bar{z}} = \sqrt{a^2 + b^2}$.

This will come in handy later.

\textbf{2.3.4 Multiplication Interpreted Trigonometrically}

Equation (2.11) showed that we can find a complex number $z$ if we know its angle $\theta$ and its distance from the complex origin $r$:

$$z = a + bi = r(\cos \theta + i \sin \theta).$$  \hfill (2.15)

Complex Number Interpreted Trigonometrically

What would happen if we multiplied the trigonometric form of two complex numbers? If $u = r(\cos \theta + i \sin \theta)$, and $v = s(\cos \phi + i \sin \phi)$, their product is

$$uv = r(\cos \theta + i \sin \theta) \cdot s(\cos \phi + i \sin \phi).$$

$$= rs(\cos(\theta + \phi) + i \sin(\theta + \phi)).$$

This is what we should have expected from the geometric interpretation. The product is a complex number whose distance from the origin is $rs$ and whose angle is $\theta + \phi$. The product of two complex numbers can be expressed in trigonometric form as $rs(\cos(\theta + \phi) + i \sin(\theta + \phi))$. This is the basis for the product rule for complex numbers:

$$uv = rs(\cos(\theta + \phi) + i \sin(\theta + \phi)).$$
Musical Signals

We can simplify this if we let \( a = \cos \theta, b = \sin \theta, c = \cos \phi, \) and \( d = \sin \phi. \) Substituting these definitions into the equation yields

\[
uv = r(a + ib) \cdot s(c + id).
\]

We can then expand it as follows:

\[
uv = rs(a + ib)(c + id) = rs(ac + aid + ibc + bdi^2).
\]

Remembering that \( i^2 = -1, \) we have

\[
uv = rs(ac + aid + ibc - bd).
\]

Grouping terms into complex number format,

\[
uv = rs[(ac - bd) + i(bc + ad)].
\]

Now if we substitute back the original terms for \( a, b, c, \) and \( d, \) we end up with

\[
uv = rs[(\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\sin \theta \cos \phi + \cos \theta \sin \phi)] \quad (2.16)
\]

Where is all this leading? I believe there’s an unwritten rule in mathematics that equations must get longer before they can get shorter. The good news is, we’ve reached the point where this one starts getting shorter. But first, there are two tools we need. A modest application of trigonometry (see appendix section A.2) demonstrates that

\[
\cos \theta \cos \phi - \sin \theta \sin \phi = \cos(\theta + \phi),
\]

and

\[
\sin \theta \cos \phi + \cos \theta \sin \phi = \sin(\theta + \phi).
\]

These trigonometric identities allow us to reduce the size of equation (2.16) substantially. Substituting these identities back into (2.16), we get:

\[
uv = rs[\cos(\theta + \phi) + i\sin(\theta + \phi)].
\]

Complex Multiplication Interpreted

Trigonometrically \quad (2.17)

Equation 2.17 tells a much simpler story about the product of \( uv \) than (2.16) does. The product of \( uv \) is scaled by the product of the magnitudes \( r \) and \( s, \) and the angle of \( uv \) is the sum of the angles \( \theta \) and \( \phi. \)

The product of two complex numbers is the product of their magnitudes and the sum of their angles.

This is much simpler and more intuitive than the algebraic definition given in equation (2.7).

Interpreting \( i \) Geometrically \quad How can we represent \( i \) itself as a complex number in trigonometric form? What are its angle and magnitude? By definition, \( i \) corresponds to the value \(+1\) on the imaginary axis (figure 2.6). So we can represent \( i \) as having magnitude \( r = 1 \) and angle \( \theta = 90^\circ. \) With
this information, and recalling equation (2.15), we can write the trigonometric form of $i$:

$$i = 1(\cos 90^\circ + i\sin 90^\circ)$$

$$= 1\left(\cos \frac{\pi}{2} + i\sin \frac{\pi}{2}\right). \quad (2.18)$$

Now, $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$. Substituting, we have

$$i = 1\left(\cos \frac{\pi}{2} + i\sin \frac{\pi}{2}\right)$$

$$= 0 + i.$$

We have demonstrated that the complex value of $i$ is $(0 + i)$, ending up right back where we started. But now we also know its trigonometric expression and have an idea of how to visualize it graphically.

**Multiplying by $i$**  What happens if we multiply $i$ times a complex number $z$? Recalling equations (2.17) and (2.18),

$$z \cdot i = r[\cos(\theta + 90^\circ) + i\sin(\theta + 90^\circ)]. \quad (2.19)$$

We just rotate $z$ counterclockwise by $90^\circ$. Suppose we set $z = 1 + 0i$, which makes it a positive vector lying on the real axis of magnitude 1. Its trigonometric form is $z = 1(\cos\theta + i\sin\theta)$ because this is also a positive vector lying on the real axis of magnitude 1. Equation (2.19) says that if we multiply $z$ times $i$, we end up with

$$z = [\cos(0 + 90^\circ) + i\sin(0 + 90^\circ)],$$

which is a positive vector lying on the imaginary axis of magnitude 1. In other words, we've just rotated $z$ by $90^\circ$ counterclockwise, leaving its length the same.
Multiplying a number by $i$ rotates it $90^\circ$ counterclockwise; its magnitude remains the same.

Note that this rule works for complex numbers, real numbers, and imaginary numbers alike. For example, if we start with 1, and multiply it by $i$, we rotate it by $90^\circ$ counterclockwise, obtaining $1.0i$. Similarly, multiplying $i \cdot i$ rotates $i$ by $90^\circ$ counterclockwise, and gives $-1$ (because $i^2 = -1$). Multiplying $-1 \cdot i$ yields $-i$, and multiplying $-i \cdot i$ gives 1 again. This is shown graphically in figure 2.7.

2.3.5 Squaring a Complex Number

Using equation (2.17), we know that $z^2$ will have an absolute value of $r^2$ and an argument of $\theta + \theta = 2\theta$:

$$z^2 = r^2 (\cos 2\theta + i \sin 2\theta).$$

To square a complex number, square the magnitude and double the angle.

2.3.6 Complex Division with Trigonometry

If multiplying two complex numbers means multiplying the magnitudes and adding the angles, it follows that division means dividing the magnitudes and subtracting the angles. The ratio of two complex numbers $u$ and $v$ can be expressed as

$$\frac{u}{v} = \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \phi + i \sin \phi)} = \frac{r_1}{r_2} [\cos (\theta - \phi) + i \sin (\theta - \phi)].$$

The angle of the argument will be rotated in the clockwise direction. What if the denominator is $i$? In that case,

$$\frac{u}{i} = \frac{r [\cos \theta + i \sin \theta]}{1 [\cos 90^\circ + i \sin 90^\circ]} = r [\cos (\theta - 90^\circ) + i \sin (\theta - 90^\circ)].$$

To divide a number by $i$, rotate it clockwise $90^\circ$; its magnitude remains unchanged.
2.3.7 Tricks with $i$

Remember these useful tricks, they will come in handy soon:

- Multiplying by $-i$ is the same as dividing by $i$. That is, $1/i = -i$. To see this, rotate a complex vector of unit length clockwise by $90^\circ$ (figure 2.7).
- $(1/i)(1/i) = (-i)(-i) = -1$. To see this, rotate a complex vector of unit length clockwise by $180^\circ$ (figure 2.7).

2.4 de Moivre’s Theorem

So far, we have been laying the groundwork for our trip up Complex Mountain, buying supplies and trekking to the edge of the foothills. In the following sections we begin to see some lofty sights. The first is de Moivre’s theorem.

The square of a complex number $z$ is

$$z^2 = r^2 (\cos 2\theta + i \sin 2\theta).$$

Multiplying again by $z$, we have $z^2 \cdot z = r^3 (\cos 3\theta + i \sin 3\theta)$, and in general,

$$z^n = r^n (\cos n\theta + i \sin n\theta), \quad n = 0, 1, 2, \ldots \tag{2.20}$$

Consider the set of all complex numbers whose magnitude is 1. They are all a unit distance from complex zero, $0 + 0i$, the origin of the complex plane, which means they form a circle around complex zero (figure 2.7). The points are complex numbers on the unit circle because they are all a unit distance from complex zero.

If we view complex numbers as vectors, complex multiplication and division are nothing more than rotating these vectors around complex zero and scaling their magnitudes.

What if the magnitudes of the two complex numbers being multiplied are both exactly unity? Since we are multiplying unities, we’d expect that all we do is spin the vector around while the magnitude remains the same, and this is indeed what happens. For example, in equation (2.20), let $r = 1$, and observe that no matter what value we assign to $n$, $z^n$ will always have a magnitude of 1, placing it always on the unit circle.

Though perhaps it is not obvious at first, we can easily rewrite the term $\cos n\theta + i \sin n\theta$ from (2.20) as $(\cos \theta + i \sin \theta)^n$. Clearly, $(\cos \theta + i \sin \theta)^n$ is a complex number with unity magnitude raised to a power. Let $z^n = (\cos \theta + i \sin \theta)^n$.

But we also know that any complex number $z^n$ can be written as $z^n = r^n (\cos n\theta + i \sin n\theta)$ for some value of $r$. And, in this case, $r = 1$ because we are looking only at points on the unit circle with unity magnitude:

$$z^n = r^n (\cos \theta + i \sin \theta)^n = r^n (\cos n\theta + i \sin n\theta)$$

$$= 1^n (\cos n\theta + i \sin n\theta).$$
Musical Signals

and since \(1^n = 1\), we have shown that
\[
\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n.
\]

**de Moivre’s Theorem:** Raising a complex number with unity magnitude to a power \(n\) is equivalent to multiplying the angle of the complex number by \(n\).

For example, if we raise a complex number with unity magnitude to increasing powers of \(n\), we set the number to spin counterclockwise around the unit circle. The greater the value of \(\theta\), the greater the angular distance covered by each increase in \(n\).

If the magnitude of the vector \(r\) is not unity, then in general,
\[
r^n(\cos n\theta + i \sin n\theta) = [r(\cos \theta + i \sin \theta)]^n.
\]

*de Moivre’s Theorem* (2.21)

This provides a simple formula for calculating powers of complex numbers. If \(z = r(\cos \theta + i \sin \theta)\), then
\[
z^n = r^n(\cos n\theta + i \sin n\theta).
\]

### 2.4.1 Taylor Series for Sine and Cosine

Brook Taylor\(^6\) knew that some series had been shown to be equivalent to trigonometric functions. For example, he knew that
\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{2.22}
\]

and
\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2.23}
\]

where \(x\) is a real number in radians. The ellipses in these equations mean these series must extend to infinity before they are equal to the expressions on the left side. The ! operator is the factorial operator in mathematics. For example, \(3! = 3 \times 2 \times 1\). In general, \(n! = n(n - 1)(n - 2) \cdots (1)\).

Note the symmetry of these equations. Both begin with positive terms, then alternate signs, +, −, +, . . . . The series for \(\sin x\) is defined using only odd numbers, and the series for \(\cos x\) is defined using only even numbers.

Many such series were developed by Brook Taylor and others because they wanted to find quick ways to approximate the numeric value of trigonometric relations. Each subsequent term in equations (2.22) and (2.23) is much smaller than the preceding one, so these series converge quickly to their target values. Hence, we can compute the sine or cosine of an angle to any degree of precision desired simply by summing more and more terms of these equations. The more terms that are summed, the more precise the result. Summing an infinite number of terms produces the value exactly. Although with modern computers this application of the Taylor series is no longer a pressing concern, (2.22) and (2.23) can be combined in a kind of Chinese puzzle that reveals a most startling result.
2.4.2 Value of $e$

What is $e$? A dictionary will indicate that it is the base of the natural logarithms, whose symbol honors the great mathematician Leonhard Euler. The first few decimal places of $e$ offer very little additional insight into its nature. They are 2.718281828459045235. . . . Like $\pi$, $e$ is an irrational constant that is useful in a variety of mathematical contexts. We can take a kind of “black box” approach to it, that is, use it without thinking too much about what it is.

Then why did I even bring up the subject of $e$? Well, it turns out that the Taylor expansion of $e$,

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \ldots$$

Taylor Expansion of $e$ (2.24)

links up with the sine and cosine series in (2.22) and (2.23) in a way that has great practical bearing. Consider this series for $e$ raised to a power $x$:

$$e^x = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

(2.25)

As with the sine and cosine series, the series in (2.25) provides a way to compute an approximate value to any desired precision of $e$ to any power $x$.

2.5 Euler’s Formula

If we substitute the complex number $z$ into equations (2.22), (2.23), and (2.25), we have

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots, \quad \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

(2.26)

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots, \quad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

(2.27)

and

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} + \ldots$$

(2.28)

The similarities in the patterns of (2.26), (2.27), and (2.28) are striking. It seems that the series for $e^z$ is made up of the sine and cosine series interleaved in some way . . . except that the sine and cosine series alternate plus and minus signs whereas the series for $e^z$ only sums positive terms. If we could find a way to relate these three series, we’d have a path toward linking $e$ to the cosine and sine functions.
Musical Signals

It looks like the expansion of $e^z$ somehow combines the terms for the expansions of $\sin z$ and $\cos z$. What if we sum the sine and cosine series just to see what they look like together?

$$\cos z + \sin z = 1 + z - \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!} - \frac{z^6}{6!} - \frac{z^7}{7!}. \quad (2.29)$$

Repeating pattern of signs

Notice that the signs in (2.29) show a repeating pattern: $++,+,−,−,++,−,−$. This feels familiar. Looking back at figure 2.7, recall that the powers of $i$ have the same periodicity of signs:

$$\begin{align*}
i^0 &= 1 & + \\
i^1 &= i & + \\
i^2 &= -1 & - \\
i^3 &= -i & - \\
i^4 &= 1 & + \\
\vdots & & \vdots
\end{align*}$$

So the series for $e$ is identical to the series for $\cos z + \sin z$ except that the terms of the $\cos z + \sin z$ series switch signs with the same periodicity as successive powers of $i$. If we modify the equation for $e^z$ to be $e^{iz}$, the effect on the right-hand side of equation (2.28) would be as follows:

$$\begin{align*}
e^{iz} &= 1 + \frac{iz}{1!} + \frac{z^2}{2!} + \frac{iz^3}{3!} + \frac{z^4}{4!} + \frac{iz^5}{5!} + \frac{z^6}{6!} + \frac{iz^7}{7!} + \cdots \\
&= 1 + \frac{iz}{1!} - \frac{z^2}{2!} - \frac{iz^3}{3!} - \frac{z^4}{4!} - \frac{iz^5}{5!} - \frac{z^6}{6!} - \frac{iz^7}{7!} - \cdots \\
&\quad + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \frac{z^8}{8!} + \cdots
\end{align*} \quad (2.30)$$

We can see the destination hidden in equation (2.30). Notice that the even exponents correspond to the series for $\cos z$, and the odd exponents correspond to the series for $\sin z$. This might be clearer if we rearrange (2.30) to group the even exponents and then group the odd exponents, which all contain $i$:

$$e^{iz} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + \left(\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right). \quad (2.31)$$

Since the left group of terms equals $\cos z$, and the right group of terms equals $\sin z$, we have shown that

$$e^{iz} = \cos z + i \sin z. \quad \text{Euler’s Formula} \quad (2.32)$$

This famous result is known today as Euler’s formula or Euler’s equation. It links the hyperbolic functions involving $e$ to trigonometric functions involving $\pi$. 
2.5.1 But Where Is \( \pi \)?

If equation (2.32) indeed links \( e \) and \( \pi \), where is \( \pi \)?

First, let's add a new variable \( a \) on both sides of equation (2.32):

\[
e^{i(z+a)} = \cos(z+a) + i\sin(z+a).
\] (2.33)

Second, recall from trigonometry that because there are \( 2\pi \) radians in a circle, \( \cos z = \cos(z + 2\pi) \) and \( \sin z = \sin(z + 2\pi) \). In general, the same is true for any integer multiple of \( 2\pi \). This means that the sine and cosine functions are periodic:

\[
\cos z = \cos(z + n2\pi),
\]

\[
\sin z = \sin(z + n2\pi),
\]

where \( n \) is any integer. As shown in figure 2.8, this makes intuitive sense if we remember that adding \( n2\pi \) to some angle \( z \) returns us to the same spot on the circle each time, so long as \( n \) is an integer.

Now, if we let \( a = 2\pi \) in equation (2.33), we have

\[
e^{i(z+2\pi)} = \cos(z + 2\pi) + i\sin(z + 2\pi).
\]

But, as we've just seen, this is identical to

\[
e^{i(z+0)} = \cos(z + 0) + i\sin(z + 0),
\]

and therefore we've shown that

\[
e^{i(z+2\pi)} = e^{iz}.
\] (2.34)

As promised, we've introduced \( \pi \) to Euler's formula. But that's not all we get for the effort. It follows that \( e^{iz} \) is periodic with period \( 2\pi \) just as the sine and cosine functions are:

\[
e^{i(z+n2\pi)} = e^{iz},
\] (2.35)

Figure 2.8
Scaling an angle by \( n2\pi \).
where \( n \) is any integer. Perhaps this is not so surprising when we remember that we have related \( e^{ix} \) to functions of circles, and circles are... well, circular.

### 2.5.2 The Most Beautiful Formula

To see more of this interesting view, let’s climb a little higher. Because \( \cos 2\pi n = 1 \) and \( \sin 2\pi n = 0 \), for integer values of \( n \), it follows that

\[
e^{i2\pi n} = \cos (2\pi n) + i \sin (2\pi n)
\]

\[= 1 + 0i,
\]

revealing that \( e^{i2\pi n} \) equals 1 regardless of the value of \( n \) (so long as \( n \) is an integer). This is a pretty startling result, actually. We are within striking distance of a truly breathtaking vista. Suppose we simplify equation (2.36) by setting \( n = 1/2 \) (relaxing the requirement that \( n \) be an integer). Substituting, we have

\[
e^{i2\pi (1/2)} = e^{i\pi} = \cos \pi + i \sin \pi,
\]

and since we know that \( \cos \pi = -1 \) and \( \sin \pi = 0 \), we have

\[
e^{i\pi} = -1 + 0i
\]

\[= -1.
\]

Finally, if we rearrange equation (2.37) slightly, we get

\[
e^{i\pi} + 1 = 0.
\]

This equation brings together five of the most important numerical values in mathematics, \( e \), \( i \), \( \pi \), 1, and 0, in one simple, elegant relation. Equation (2.38) has been described as the most beautiful formula in mathematics. It’s like seeing the entire panoply of the planets together with a crescent moon at sunset. By analogy, this equation can also be seen to integrate the four main branches of mathematics: 0 and 1 from arithmetic, \( \pi \) from geometry, \( i \) from algebra, and \( e \) from analysis.

Equation (2.38) is the cornerstone of a major mathematical edifice that represents musical signals, among other things, in a crisp and penetrating way. This goes to show, as Sir D’Arcy Wentworth Thompson (1917) wrote, “The perfection of mathematical beauty is such... that whatsoever is most beautiful and regular is also found to be most useful and excellent.”

### 2.5.3 What Is \( e^i \) by Itself?

We got such a fine result by introducing \( \pi \) into Euler’s identity that it seems a shame to remove it, but to find out the value of \( e^i \) by itself, we go back to the Taylor series definition of \( e \) raised to a power, given in equation (2.25):

\[
e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots.
\]
We want to find the solution to this equation for \( x = 0 + 1i \). Substituting, we get

\[
e^{0+1i} = 1 + \frac{i}{1!} + \frac{i^2}{2!} + \frac{i^3}{3!} + \frac{i^4}{4!} + \frac{i^5}{5!} + \cdots
\]

\[
= 1 + i\frac{1}{2!} - \frac{i}{3!} + \frac{1}{4!} + i\frac{1}{5!} - \cdots
\]

Grouping even and odd terms yields

\[
e^i = 1 - \frac{1}{2!} + \frac{1}{4!} - \frac{1}{6!} + \cdots + i\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots\right)
\]

which says that \( e^i \) is a complex number with a real part equal to \( \cos 1 \) and an imaginary part equal to \( \sin 1 \). Remembering that the arguments to the sine and cosine functions are in units of radians, we can write

\[
e^i = (\cos 1 + i \sin 1),
\]

which allows us further to say that equation (2.40) represents a vector in the complex plane of unit length and angle of 1 radian. The Cartesian values are approximately \( x = \cos 1 \approx 0.543 \) and \( y = \sin 1 \approx 0.841 \).

### 2.6 Phasors

Consider the complex number \( z = x + iy \). If we let \( x = \cos \theta \) and \( y = \sin \theta \), then

\[
z = \cos \theta + isin \theta,
\]

and we know by Euler’s formula that

\[
\cos \theta + i \sin \theta = e^{i\theta}.
\]

(2.41)

Now, if we set \( \theta = 1 \), we’ve simplified back to equation (2.40), where we established that by itself \( e^i \) can be thought of as a vector from the origin of the complex plane with length 1 and angle of 1 radian. But let’s leave \( \theta \) in the equation and assume \( \theta \) represents a real number.

Suppose \( \theta \) gradually decreases from 1 to 0. At first, when \( \theta = 1 \), the value will be \( e^{i1} = \cos 1 + isin 1 \), just as in equation (2.41). But as \( \theta \) decreases, \( e^{i\theta} \) rotates clockwise (figure 2.9). When \( \theta \) gets to 0, we have

\[
e^{i0} = \cos 0 + i \sin 0
\]

\[
= 1 + i0,
\]
Figure 2.9
Unit vector as θ decreases toward zero.

that is, the vector $e^{i\theta}$ will be lying along the positive real axis:

$$0 \quad e^{i\theta} \quad 1$$

Let’s check out some other interesting values for θ. We’ve already seen by equation (2.38) that when $\theta = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi$$
$$= -1 + 0i,$$

which means the vector $e^{i\theta}$ will be lying along the negative real axis:

$$-1 \quad e^{i\pi} \quad 0$$

Two other values of θ are noteworthy. If we set $\theta = \pi/2$:

$$e^{i(\pi/2)} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$$
$$= 0 + 1i,$$

which means the vector $e^{i\theta}$ will be lying along the positive imaginary axis. And if we set $\theta = 3\pi/2$,

$$e^{i(3\pi/2)} = \cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2}$$
$$= 0 - 1i,$$

which means the vector $e^{i\theta}$ will be lying along the negative imaginary axis.
Finally, if $\theta = 2\pi$,

\[ e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1 + 0i, \]

which means the vector $e^{i\theta}$ has gone back to lying along the positive real axis, making one complete rotation.

Thus, as $\theta$ goes from 0 to $2\pi$, the vector $e^{i\theta}$ spins once counterclockwise on its axis in the complex plane. This is shown in Figure 2.10. One complete rotation is called a period. The different positions the vector reaches on the unit circle during a period are referred to as its phases. Since the angle $\theta$ controls the phase, it is called the phase angle.

To summarize, $e^i$ is a unit vector, that is, a vector of length 1 and angle of 1 radian. For $e^{i\theta}$, as $\theta$ goes from 0 to $2\pi$, the unit vector visits every point on the complex unit circle, including +1, i, −1, and −i. As $\theta$ increases past $2\pi$, $e^{i\theta}$ will just continue to spin, returning to +1 whenever $\theta$ is an integer multiple of $2\pi$.

Notice how much more compact it is to write $e^{i\theta}$ rather than $\cos \theta + i \sin \theta$. This provides tremendous economy of expression for discussing wave motion later.

There's only one thing missing: a way to make the vector other than unit length. If we scaled $e^{i\theta}$ by a real variable $r$, we could change the length of the vector as well:

\[ z = re^{i\theta}. \]  

(Phasor) (2.46)

As $r$ changes, the vector’s length changes, and as $\theta$ changes, the vector’s direction changes. Equation (2.46) is convenient polar representation of any complex variable. It is easier to write $z = re^{i\theta}$ than to write $z = r(\cos \theta + i \sin \theta)$, and we get the intuitive advantage of visualizing $z$ as a vector spinning around the origin of the complex plane. Polar representation of complex variables is so powerful that it has its own name: equation (2.46) is called the phasor.
Musical Signals

The two variables $r$ and $\theta$ in (2.46) allow us to identify uniquely any point on the complex plane. Imagine a line of length $r$ called the radial coordinate with its base anchored at the origin of the complex plane. The counterclockwise angle of this line above the real axis is given by $\theta$, called the angular coordinate, or polar angle. Together, $r$ and $\theta$ are called the polar coordinates. They are related to Cartesian coordinates by

$$x = r \cos \theta, \quad \text{and} \quad y = r \sin \theta,$$

where $r$ is the radial distance from the origin, and $\theta$ is the angle from the real axis traveling counterclockwise. In terms of $x$ and $y$, by the Pythagorean theorem,

$$r = \sqrt{x^2 + y^2},$$
$$\theta = \tan^{-1} \frac{y}{x}.$$

2.6.1 Circular Motion

In the previous section, we scaled Euler's formula by a real variable $r$,

$$re^{i\theta} = r(\cos \theta + i \sin \theta),$$

defined as a phasor of magnitude $r$. Figure 2.9 showed that as $\theta$ decreases, the phasor spins clockwise, and figure 2.10 showed that as $\theta$ increases, the phasor spins counterclockwise.

2.6.2 The Upside-Down Bicycle

Imagine a phasor as a spoke on a bicycle wheel that has been painted red. It spins one way or another depending on whether $\theta$ is increasing (becoming more positive) or decreasing (becoming more negative). The variable $r$ can be thought of as the length of the spoke and hence the size of the wheel. One spoke on the other wheel is also painted red so we can track them both easily. Now spin one wheel counterclockwise, corresponding to $\theta$ growing more positive, and spin the other clockwise, corresponding to $\theta$ growing more negative.

As the wheels spin, how many revolutions do the wheels make per second? Suppose we observe that the red spokes on both wheels rotate once every second, or 1 Hz. We can relate the speed of rotation to the rate at which the angle of the red spoke on each wheel changes. Since the spokes are rotating at 1 Hz, the angle of each red spoke travels through $2\pi$ radians per second. The faster they spin, the higher the frequency and the greater the angular velocity.

Now close your eyes and quickly blink them open and shut, noting the phase angles of the red spokes on the two wheels. If you blink quickly enough, the spokes seem not to be moving. You have observed the instantaneous phase angle of the spokes.

Now stop the bicycle wheels and spin one, and then the other, clockwise. Observe the instantaneous phase angles again. Since one wheel started after the other, their instantaneous phase angles will not be equal even if they rotate with the same frequency: one wheel will lead the other by some amount. This is their phase difference.
If the wheels travel at the same exact frequency, the phase difference will be a constant. If, as is more likely, one wheel travels faster than the other, the phase difference will change gradually as the wheels turn, and one red spoke will overtake the other and then pass it. One wheel is said to be precessing the other. The rate of precession is the time it takes for the phase difference between them to return to the initial phase difference.

Again, spin one wheel clockwise and the other counterclockwise so that they are traveling at the same rate of speed in opposite rotation. Even if they travel at exactly the same speed, the angular velocities of the two wheels are not equal because the radian velocity of the wheel turning counterclockwise is $2\pi$ radians per second, whereas the radian velocity of the wheel turning clockwise is $-2\pi$ radians per second. Thus, if the wheel turning counterclockwise with positive radial velocity has a frequency of 1 Hz, the one turning clockwise with negative radial velocity must have a frequency of $-1$ Hz. Positive frequencies correspond to counterclockwise rotation and negative frequencies correspond to clockwise rotation.

### 2.6.3 Positive and Negative Frequencies

How can we express positive and negative radian velocity mathematically? We can understand positive frequencies with the phasor

$$re^{i\theta} = r(\cos \theta + i \sin \theta),$$  \hspace{1cm} (2.47)

such that as $\theta$ increases, the phasor spins counterclockwise, corresponding to positive frequencies.

Can we represent negative frequencies simply by inverting the sign of $\theta$? That is, what about $e^{-i\theta} = e^{-i\theta}$? It should work, but let’s check. Going back to Euler’s formula, we see that if $e^{i\theta} = \cos \theta + i \sin \theta$, then

$$e^{-i\theta} = (\cos -\theta) + (i \sin -\theta)$$

$$= \cos \theta - i \sin \theta$$  \hspace{1cm} (2.48)

because $\cos -\theta = \cos \theta$, and $\sin -\theta = -\sin \theta$.

By (2.48), as $\theta$ begins to increase from 0, the real part, $\cos \theta$, will go from 1 toward 0 (shrinking along the x-axis toward 0) while the imaginary part, $-i \sin \theta$, will go from 0 toward $-i$ (growing along the negative y-axis). When $\theta = 0$, the negative phasor $e^{-i\theta}$ starts off lying along the real axis line just like its positive cousin $e^{i\theta}$. But as $\theta$ begins to increase from 0, the negative phasor drops down and to the left, beginning a clockwise rotation, just as we wanted.

Thus, we can use the negative-frequency phasor $e^{-i\theta}$ to represent negative, clockwise-turning frequencies and the positive-frequency phasor $e^{i\theta}$ to represent positive, counterclockwise-turning frequencies. This is summarized in Figure 2.11.

### 2.6.4 Complex Harmonic Motion

We know that simple harmonic motion is the projection of circular motion onto one-dimensional displacement (see volume 1, section 5.1, especially volume 1, figure 5.7). And sinusoidal motion
Positive-Frequency Phasor: 
\[ e^{i\theta} = \cos \theta + i \sin \theta \]

Negative-Frequency Phasor: 
\[ e^{-i\theta} = \cos \theta - i \sin \theta \]

For increasing \( \theta \)

**Figure 2.11**
Positive- and negative-frequency phasors.

**Figure 2.12**
Projecting sine and cosine from the same circular motion.

is the projection of simple harmonic motion through time (see volume 1, figure 5.9). The root motion governing both is circular motion.

Figure 2.12 shows two spotlights at right angles to each other projecting the shadow of a cone mounted on a turntable onto two screens. As the turntable turns, the harmonic motion of the shadows projected on the screens will show a phase difference of 90°, precisely the phase difference between sine and cosine. If we think of the cone on the turntable as a phasor on the complex plane, then the shadows describe the motion of the projected sine and cosine harmonic motions. The only difference between sine and cosine is the angle of projection.

If we look at figure 2.12 from directly overhead, as shown in figure 2.13, we see that the phasor \( e^{i\theta} \) does indeed embody both the cosine and sine relations simultaneously. In this graph the circle
is plotted in the complex plane, but the projected sine and cosine waves are plotted as real values of amplitude against angle $\theta$ as it moves in time from 0 to $2\pi$.

Project along the real axis in figure 2.13 as $\theta$ goes from 0 to $2\pi$. When $\theta = 0$, the phasor lies to the right of complex zero along the real axis, and it swings around counterclockwise as $\theta$ increases. We see that the phasor $e^{i\theta}$ makes a full circle counterclockwise, beginning and ending at $e^{i0} = 1 + 0i$. As the phasor turns, we see by inspection that the projected point $\sin \theta$ describes a sine wave because it begins at amplitude 0, gradually increases in amplitude to 1, works its way back to 0, then to $-1$, and finally returns to 0, just as a sine wave does. Similarly, the projected point $\cos \theta$ describes a cosine wave because it begins at amplitude 1, decreases in amplitude to $-1$, then works its way back to 1.

When signals are lock-stepped at a $90^\circ$ phase difference like the sine and cosine projections of circular motion, the signals are said to be in quadrature. Although quadrature has a number of meanings in mathematics, in this context it means a $90^\circ$ phase relation between two periodic quantities varying with the same period.
2.6.5 Sinusoids

Looking at figure 2.13, we see that if we project vertically across the real axis, the phasor \( e^{i\theta} \) generates \( \cos \theta \) as \( \theta \) increases, and if we project horizontally across the imaginary axis, the phasor generates \( \sin \theta \) as \( \theta \) increases. Recalling Euler's formula, figure 2.13 shows that a phasor is indeed the sum of a cosine and a sine in quadrature because they both emerge as projections of \( e^{i\theta} = \cos \theta + i\sin \theta \).

But why restrict ourselves to projecting just along the two dimensions of the complex plane? We can swivel the projector around to any arbitrary angle and create a host of different but related wave functions. Swiveling the light to 45°, we produce a wave that is halfway between a sine and cosine wave (figure 2.14). The formula for this wave is

\[
\cos(\theta + 45^\circ) + i\sin(\theta + 45^\circ).
\]

As the projector is swiveled around an entire circle, we observe all possible combinations of sine and cosine wave. A slight modification of Euler's formula allows us to represent this process of projecting from different angles:

\[
e^{i(\theta + \phi)} = \cos(\theta + \phi) + i\sin(\theta + \phi),
\]

\(\text{Sinusoids } (2.49)\)

Figure 2.14
Phasor projected at 45°.
where \( \phi \) is the phase angle of the projector. The family of curves defined by (2.49) are the sinusoids. By allowing the projected angle to vary, we allow all projections through a phasor from all possible angles.

### 2.6.6 Mixing Sine and Cosine to Create Sinusoids

Another way to create a sinusoid of any phase does not require complex arithmetic: we add a sine and a cosine together, varying the strengths of each to get the desired phase offset. For example, \( a \cos \theta + b \sin \theta \), with \( a = 1 \), \( b = 0 \), reduces to \( \cos \theta \); and with \( a = 0 \), \( b = 1 \), to \( \sin \theta \); and with \( a = b = 1 \), to an equal mixture of the two. What does that look like? Plot sine and cosine waves of equal amplitude and sum them point by point (figure 2.15). Note the resemblance of this shape to figure 2.14.

We have shown experimentally that \( \cos \theta + \sin \theta = \sin (\theta + 45^\circ) \). Using trigonometry, we can generalize this to show that

\[
M \sin(\theta + \phi) = a \cos \theta + b \sin \theta
\]

for appropriate choices of \( a \), \( b \), and \( \phi \).

### 2.6.7 Positive and Negative Frequencies and Amplitudes

Recall from section 2.6.1 that frequencies are positive or negative depending upon the direction of their circular motion. As shown in figure 2.11, a negative-frequency phasor \( e^{-i\theta} \) turns clockwise, and a positive-frequency phasor \( e^{i\theta} \) turns counterclockwise. Both these phasors have positive magnitudes. What is the behavior of negative-amplitude phasors \( -e^{i\theta} \) and \( -e^{-i\theta} \)?

![Figure 2.15](image)

*Sum of cosine and sine.*
A negative-amplitude, positive-frequency phasor $-e^{i\theta}$ still generates a positive frequency because it still turns counterclockwise as $\theta$ increases. However, it has negative length, which means that it points in the opposite direction to a corresponding positive-length phasor. Consider the phasor $e^{i\theta} = 1 + 0i$, which is a unit vector lying on the real axis, anchored at complex zero and pointing to the right along the real axis:

\[ e^{i\theta} \]

If we negate it, we have $-e^{i\theta} = -1 + 0i$, which points to the left along the real axis:

\[ -e^{i\theta} \]

Recall from equation (2.42) that $e^{i\pi} = -1 + 0i$, so $-e^{i\theta} = e^{i\pi}$. Think of it this way: if we start out with $e^{i\theta}$, a real unit vector pointing to the right, there are two ways we can make it point in the opposite direction: we can rotate it around complex zero (clockwise or counterclockwise) by a half-circle, $e^{i\pi}$, or we can negate it, $-e^{i\theta}$.

Similarly, if a negative-frequency phasor $e^{-i\theta}$ is negated, it becomes $-e^{-i\theta}$. It still has a negative frequency because it turns clockwise as $\theta$ increases.

If we want to reverse the direction of a negative-frequency phasor, we have the same two choices as with the positive-frequency phasor: we can either rotate the phasor a half-circle around complex zero or flip its direction by negation.

These concepts are shown graphically in figure 2.16. A positive-frequency phasor with negative amplitude is identical to that same positive-frequency phasor rotated forward or backward by $\pi$ radians (180°). In other words, $-e^{i\theta} = e^{i\theta + \pi}$. Similarly, $-e^{-i\theta} = e^{-i\theta + \pi}$.

Another way to view this is to look directly at the sine and cosine of $\theta$ when we add $\pi$ to it (figure 2.17). We can see by inspection for sine and cosine that adding $\pi$ to any angle is the same as negating it.

### 2.6.8 Phasors and Sinusoids

What would happen if we took a positive-frequency phasor $e^{i\theta} = \cos \theta + i \sin \theta$ spinning counterclockwise and a negative-frequency phasor $e^{-i\theta} = (\cos -\theta) + (i \sin -\theta)$ spinning clockwise and added them together?

When a positive-frequency phasor and a negative-frequency phasor are tied to the same angular displacement, they are in conjugate symmetry, so

\[ e^{i\theta} + e^{-i\theta} = (\cos \theta) + (i \sin \theta) + (\cos -\theta) + (i \sin -\theta) \]

\[ = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta \]

\[ = 2 \cos \theta. \]

\[ e^{i\theta} \]

\[ e^{-i\theta} \]

\[ -e^{i\theta} \]

\[ -e^{-i\theta} \]
Positive Complex Frequency
with positive amplitude:
\[ e^{i\theta} = \cos \theta + i\sin \theta \]
with negative amplitude:
\[ -e^{i\theta} = -(\cos \theta + i\sin \theta) \]

Negative Complex Frequency
with positive amplitude:
\[ e^{-i\theta} = \cos \theta - i\sin \theta \]
with negative amplitude:
\[ -e^{-i\theta} = -(\cos \theta - i\sin \theta) \]

For increasing \( \theta \)

![Diagram showing positive and negative frequencies and amplitudes.](image)

**Figure 2.16**
Positive and negative frequencies and amplitudes.

![Diagram showing magnitudes are the same.](image)

**Figure 2.17**
Adding \( \pi \) to an angle \( \theta \).

The result takes a bit of explaining, but it will prove to be crucial information. Rearranging (2.51) slightly, we get

\[
\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{e^{i\theta}}{2} + \frac{e^{-i\theta}}{2}.
\]

*Real Cosine as Sum of Two Phasors in Conjugate Symmetry* (2.52)

That's a real cosine on the left side of equation (2.52). We've already seen a couple of other ways to create cosine waves, but this formula is the root of all other explanations. When summed, the real parts of phasors in conjugate symmetry add constructively and the imaginary parts cancel (figure 2.18). The sum of phasors in conjugate symmetry is always on the real number line. As \( \theta \) varies, the length of the sum vector varies as \( \cos \theta \).
A real cosine consists of the vector sum of two half-amplitude phasors of opposite frequency.

Now, what if we subtract two phasors in conjugate symmetry?

\[
e^{i\theta} - e^{-i\theta} = (\cos \theta + i \sin \theta) - (\cos \theta - i \sin \theta)
\]

\[
= 2i \sin \theta.
\]

Rearranging (2.53) slightly, we get

\[
\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.
\]

Real Sine as Difference of Two Phasors in Conjugate Symmetry (2.54)

To get \(i\) out of the denominator, since \(1/i = -i\), we can flip the sign of \(i\):

\[
\sin \theta = \frac{-i e^{i\theta} - e^{-i\theta}}{2}.
\] (2.55)

This result might seem counterintuitive. How can a real sine wave be made of entirely imaginary components? When subtracted, the imaginary parts of phasors in conjugate symmetry add constructively, whereas the real parts cancel (figure 2.19). If we then multiply this imaginary difference by \(-i\), we rotate it 90° clockwise to the real number line to obtain a real sinusoid. As \(\theta\) varies, this difference varies as \(\sin \theta\).

A real sine consists of the vector difference of two half-amplitude phasors of opposite frequency and amplitude.
Equations (2.52) and (2.54) are really just variations on Euler’s formula. They are the foundation of a great deal of important modern music technology. They are so important, in fact, that I present another visualization.

2.6.9 Cosine Machine

Figure 2.20 is a visual aid for equation (2.52) that I call the cosine machine. It forms the vector sum of two phasors mechanically. It has a motor with an arm attached to its rotor. At the end of the first arm is another arm of equal length connected to the first with a bearing. The end of the second arm slides from side to side in a slot. Figure 2.21 shows the cosine machine’s stages of movement. Figure 2.22 shows the operation of the cosine machine with a pen attached, producing a cosine wave.

The length of each rotating arm is 1/2, so when the bars lie flat along the slot, they add up to 1. The arm attached to the motor turns counterclockwise (θ), and the outer arm turns clockwise (−θ). The total side-to-side excursion of the arms is 2, and this motion outlines cosine movement as θ goes from 0 to 2π.

2.6.10 Sine Machine

Figure 2.23 shows an interpretation of equation (2.55) that I call the sine machine. This machine forms the vector difference of two phasors. We can split equation (2.55) into two phasors as follows:

\[ \sin \theta = -i \frac{e^{i\theta} - e^{-i\theta}}{2} = i \frac{e^{-i\theta}}{2} - i \frac{e^{i\theta}}{2}. \]
The black arm in that figure spins clockwise and is associated with the negative-frequency phasor $e^{-i\theta/2}$; the white arm spins counterclockwise, matching the positive-frequency phasor $e^{i\theta}/2$. The black arm is connected at one end to a fixed bearing mounted behind the center of the slot that the black arm swirls around. The motor housing is attached to the other end of the black arm. The motor shaft is attached to the white arm. The other end of the white arm is attached to the slot via a pin to hold it in the slot. A pen is attached to the white arm in such a way that it always points straight down.

When the motor starts, both arms are directly above the center of the slot, corresponding to $\theta = 0$. As $\theta$ increases, the white arm moves clockwise, and the black arm moves counterclockwise. The vector difference generates a sine wave as $\theta$ goes from 0 to $2\pi$. 

Figure 2.21
Stages of movement of the cosine machine.
Figure 2.22
Rotating armature generating a cosine wave.

Figure 2.23
Sine machine.
Note that the cosine machine (figure 2.22) could have been used to create a sine wave by lifting the pen until $\theta = 90^\circ$, then dropping it, introducing a phase delay. We could do exactly the same thing with the sine machine (figure 2.23) to generate a cosine wave. But the figures illustrate how to generate sine and cosine waves directly and in phase. For $0 \leq \theta \leq 2\pi$, the cosine machine (figure 2.22) generates a cosine wave, and the sine machine (figure 2.23) generates a sine wave.

### 2.6.11 Energy of a Phasor

$$E = \sqrt{2}$$

Kinetic energy is proportional to the square of velocity (see volume 1, equation (4.28)). In terms of real waveforms, energy is the square of amplitude. But what corresponds to the energy of a phasor?

If the magnitude of the phasor $z = re^{i\theta}$ is $r$, then by equation (2.14), $r^2 = zz^*$, and

$$r = \sqrt{zz^*} = \sqrt{a^2 + b^2}.$$ 

Thus, if we associate the magnitude $r$ of the phasor with the amplitude of a wave, $r^2$ is its energy.

### 2.6.12 Even and Odd Functions

Notice that the shape of the cosine wave is symmetrical around $x = 0$. The cosine function has the same value for both positive and negative $x$ indexes. That is, $\cos x = \cos (-x)$, as shown in figure 2.24. Because of this, the cosine function is an even function. In general, a function $f$ is even if $f(x) = f(-x)$.

The shape of the sine wave is antisymmetrical around $x = 0$. If we negate the sine function, we have $-f(x)$, shown as a bold curve in figure 2.25. For any $x$, the positive and negative functions are equal, and $\sin x = -\sin (-x)$. Because of this, the sine function is an odd function. In general, a function $f$ is odd if $f(x) = -f(-x)$. Table 2.3 summarizes these observations.

With the exception of the zero function, $f(x) = 0$, all functions are either even or odd, or a mixture of the two:

$$f(x) = f_e(x) + f_o(x).$$

![Cosine as an even function.](image)
Given the definitions for even and odd functions in table 2.3 and equation (2.56), it follows that:

\[ f(-x) = f_e(-x) + f_o(-x) \]

\[ = f_e(x) - f_o(x). \]  \hspace{1cm} (2.57)

Equation (2.56) says that a function of a positive index \( x \) is the sum of its even and odd parts. Equation (2.57) extends this slightly to say that a function of a negative index \( x \) is equal to the difference of its even and odd parts. If we add (2.56) and (2.57),

\[ f(x) + f(-x) = f_e(x) + f_o(x) + f_o(x) - f_o(x) \]

\[ = 2f_e(x), \]

and rearrange to solve for \( f_e(x) \), we see that

\[ f_e(x) = \frac{1}{2}[f(x) + f(-x)]. \]  \hspace{1cm} (2.58)

Equation (2.58) shows how to extract the even portion of any function \( f(x) \): compute 1/2 times the value \( f(x) + f(-x) \), and the result will be the even portion of the function.

If we subtract equation (2.57) from equation (2.56),

\[ f(x) - f(-x) = f_o(x) + f_e(-x) - f_e(x) + f_o(x) \]

\[ = 2f_o(x) \]
Musical Signals

and rearrange to solve for \( f_o(x) \), we see that

\[
f_o(x) = \frac{1}{2} [f(x) - f(-x)].
\] (2.59)

Equation (2.59) shows how to extract the odd portion of any function \( f(x) \): compute 1/2 times the value \( f(x) - f(-x) \), and the result will be the odd portion of the function.

Now, how can we be sure that equations (2.58) and (2.59) fully represent the whole of the function \( f(x) \)? Well, if we add them together, we should get back the original function, which we do:

\[
f_e(x) + f_o(x) = \frac{1}{2} [f(x) + f(-x)] + \frac{1}{2} [f(x) - f(-x)] = f(x).
\]

We have proved that equations (2.58) and (2.59) fully represent the function \( f(x) \), unrestricted in any way.

**All functions can be broken down into even and odd parts using equations (2.58) and (2.59).**

In particular, if we define \( f(\theta) = e^{i\theta} \), we immediately demonstrate, as in equation (2.52), that

\[
f_e(\theta) = \frac{1}{2} (e^{i\theta} + e^{-i\theta}),
\]

and also demonstrate, as in equation (2.54), that

\[
f_o(\theta) = \frac{1}{2} (e^{i\theta} - e^{-i\theta}).
\]

Understanding even and odd functions will come in very handy in chapter 3.

2.6.13 Making Phasors Spin in Time

In figure 2.13 we saw that the phasor \( e^{i\theta} \) makes one complete period as \( \theta \) goes from 0 to \( 2\pi \). If we wish to specify that this period should occur over a particular duration \( T \), we can let \( \theta = 2\pi f/T \), where \( t \) is time. For example, if we wish the phasor to complete one period in 1 second, we set \( T = 1 \). Then, as the real-valued time parameter \( t \) goes from 0 to 1 second, the phasor \( e^{i2\pi f/T} \) goes through \( 2\pi \) radians, one full rotation.

Making a phasor spin at a particular rate puts it into the temporal realm. To signify this, let’s say that the phasor with time \( t \) in its exponent is the complex sinusoid.

What if we want the complex sinusoid to go through two periods in 1 second? The most convenient approach would be to introduce a frequency variable \( f \), so that \( \theta = 2\pi ft/T \). Now the complex sinusoid is defined as

\[
e^{i\theta} = e^{i2\pi ft/T}.
\] (2.60)
If we set \( f = 2 \), then the phasor \( e^{i2\pi f/T} \) will make two cycles in 1 second. If we set \( f = 440 \), it will make 440 cycles per second. Defining \( \theta = 2\pi f/T \) causes \( \theta \) to express angular velocity, the rate at which the phasor spins.

It is common to simplify equation (2.60) by defining \( \omega = 2\pi f \), so that the phasor becomes \( e^{i\omega T} \). Since most often \( T = 1 \) (because we’re mostly measuring frequency in Hz, which is cycles per second), we can simplify a bit further by leaving out \( T \). The time-based phasor is then simply \( e^{i\omega t} \). If we want amplitude to be other than unity, we can add an amplitude term \( A \) to scale the phasor’s magnitude, so that we have the following canonical representation for the complex sinusoid:

\[
 Ae^{i\omega t} = Ae^{i2\pi ft}. \tag{2.61}
\]

This powerful but economical representation of the complex sinusoid is used throughout the rest of the book.

### 2.7 Graphing Complex Signals

Suppose a bright light were mounted on the tip of an airplane’s propeller as it flies past at night. The forward circular motion of the light would create a helix as it cuts through the air. To represent the complex sinusoid graphically (figure 2.26a), we use a 3-D representation where the \( y \)-axis is the imaginary number line, the \( x \)-axis (shown sloping up to the right) is the real number line, and the \( z \)-axis (sloping down to the right) is time. Figure 2.26b shows a complex sinusoid as a helix. We can project along the real axis to see just the sine wave component (shown on the “wall” behind the helix), or project along the imaginary axis to see just the cosine wave motion (shown on the “floor” below the helix).

![Figure 2.26](image_url)

**Projection of a complex signal.**
Musical Signals

If the equation for the helix is \( s(t) = e^{i\omega t} \), then the cosine projection is just the real part, denoted \( \text{Re}\{s(t)\} \), and the sine projection is just the imaginary part, denoted \( \text{Im}\{s(t)\} \). The value of the helix when \( t = 0 \) in figure 2.26 is

\[
s(0) = e^{i\omega 0} = \cos 0 + i \sin 0,
\]

which in Cartesian coordinates corresponds to the 3-D point

\[
(x, y, z) = (\text{Re}\{s(t)\}, \text{Im}\{s(t)\}, t))
\]

\[
= (\cos 0, \sin 0, 0)
\]

\[
= (1, 0, 0).
\]

The point on the helix for any other time can be similarly determined by plugging in the appropriate values of \( \theta \) and \( t \). The helix in figure 2.26 is spiraling counterclockwise, indicating positive frequency.

2.8 Spectra of Complex Sampled Signals

In section 1.3.3 we established an equivalence for sampled waveforms between sampling rate and angular velocity, saying that frequency \( f \) is to the Nyquist barrier \( \pi \) as angular velocity \( \theta \) is to \( \pi \). Therefore, when plotting a spectrogram, we can simply plot frequency as radian velocity between \(-\pi\) and \(\pi\), and leave sampling rate out of it entirely. This is an advantage when comparing spectra that were sampled at different rates.

In section 2.6.7 we saw that there are positive and negative frequencies and amplitudes. As was shown in figure 2.11, the negative phasor \( e^{-i\theta} \) turns clockwise, producing negative frequencies, and the positive phasor \( e^{i\theta} \) turns counterclockwise, producing positive frequencies. Negating the sign of a phasor is the same as giving it a 180° phase shift. So, for some radian velocity \( \theta \), there are four possible phasors:

- \( e^{i\theta} \) Positive frequency, positive amplitude
- \( e^{-i\theta} \) Positive frequency, negative amplitude
- \( -e^{i\theta} \) Negative frequency, positive amplitude
- \( -e^{-i\theta} \) Negative frequency, negative amplitude

If we want to show the spectrum of a complex signal, we must find a way to represent each of these phasors distinctly. Figure 2.27 shows a complex spectrum with one of each kind of phasor. The positive-amplitude phasors are shown with bold arrows, and the corresponding negative-amplitude phasors are shown shaded. Positive and negative amplitude is graphed on the y-axis. The x-axis shows angular velocity between \(-\pi\) and \(\pi\) instead of frequency.
2.8.1 Complex Spectrum of a Real Sampled Cosine Wave

In equation (2.52) we found that \( \cos \theta = (e^{i\theta} + e^{-i\theta})/2 \), which says that a real cosine equals the sum of two half-amplitude phasors of opposite frequency. Now that we have a way of representing complex spectral components mathematically, we can diagram spectrograms of these relations.

The complex spectrum of equation (2.52) is shown in figure 2.28. The Nyquist barrier is shown as \( \pm \pi \), so the frequency \( \theta \) corresponds to \( \pm \theta / \pi \). The distance of the two arrows from zero along the x-axis represents the frequency of the two phasors: the left one has negative frequency, the right one has positive frequency. Both arrows point up because the signs of both phasors are positive, and they each have a magnitude of 0.5.

2.8.2 Complex Spectrum of a Real Sampled Sine Wave

The complex spectrum of the sine wave

\[
\sin \theta = -i\frac{e^{i\theta} - e^{-i\theta}}{2} = i\frac{e^{-i\theta}}{2} - i\frac{e^{i\theta}}{2}
\]

\[
= i\left(\frac{e^{-i\theta}}{2} - \frac{e^{i\theta}}{2}\right)
\]

is shown in figure 2.29. Both components are imaginary. This graphical representation does not allow us to show real and imaginary components together. Later in this chapter, I develop a 3-D representation of complex spectra that does. We see that real sine wave \( \sin \theta \) is made of two imaginary
Figure 2.29
Complex spectrum of a real sine wave.

half-amplitude phasors, one with positive frequency and negative amplitude, the other with negative frequency and positive amplitude.

2.9 Multiplying Phasors

In section 2.3.4 we observed that to multiply two complex numbers, we multiply their magnitudes and add their angles. Similarly, to multiply two phasors, we multiply their magnitudes and add their angles. For complex sinusoids (that is, phasors containing time \( t \) in the exponent), we must multiply their magnitudes and add their angles at every point in time.

What happens if we multiply the time-based phasor \( e^{j\omega t} \) by itself (thereby squaring it)? Recall that unless a scaling term is added the magnitude of a phasor is unity (that is, \( |e^{j\omega}| = 1 \)), so multiplying \( e^{j\omega t} \) by itself won’t change its magnitude. But its frequency will double because we sum the angles at each point in time, doubling the rotational velocity: \((\omega + \omega = 2\omega)\). Figure 2.30 shows the rotation of phasors through time as a helix. Figures 2.30a and 2.30b are identical phasors, so multiplying them is effectively a squaring operation. Figure 2.30c shows the product signal, which spins twice as fast but has the same amplitude. Multiplying phasors to raise their frequency is modulation.

If we multiply a positive-frequency phasor \( e^{j\omega t} \) by a phasor of equal but negative frequency \( e^{-j\omega t} \), the magnitude of the product will be unity, but the frequency will be 0 Hz, because \( \omega - \omega = 0 \). The result is a signal that has the value of complex unity \((1 + 0j)\) at all points (figure 2.31). Multiplying phasors to lower their frequency is demodulation.

We can change the frequency of a phasor by an arbitrary amount. Say we have two signals, \( s_1 = A_1 e^{j2\pi f_1 t} \) and \( s_2 = A_2 e^{j2\pi f_2 t} \). Their product is a signal with magnitude \( A_1 \cdot A_2 \) and frequency \( f_1 + f_2 \). For instance, if \( f_1 = 4 \text{ Hz} \) and \( f_2 = -3 \text{ Hz} \), the product will be a phasor at 1 Hz with magnitude \( A_1 \cdot A_2 \) (figure 2.32).

Modulation and demodulation are used, for example, to convert between audio-frequency signals and radio-frequency signals. For instance, suppose a radio receiver detects a signal \( f_1 = 1 \text{ MHz} \). If the receiver has an internal oscillator tuned to \( f_2 = 0.999 \text{ MHz} \) and multiplies these two signals together, the result would be an audio-frequency tone of 1 kHz.
Figure 2.30
Squaring a phasor.

\[ e^{i\omega t} \times e^{i\omega t} = (e^{i\omega t})^2 \]

Figure 2.31
Multiplying identical positive- and negative-frequency phasors.

\[ e^{i\omega t} \times e^{-i\omega t} = 1 + 0i \]

Figure 2.32
Changing the frequency of a phasor.

\[ e^{i2\pi ft} \times e^{-i2\pi ft} = e^{i2\pi f} \]
Chapter 2

Musical Signals

By far the most important musical application of modulation and demodulation is frequency
detection. The Fourier transform operates somewhat like a radio receiver to tune in and register the
frequencies present in a signal (see chapter 3).

Recall equation (2.52), which shows that a real cosine waveform is the sum of two half-amplitude
phasors with opposite frequencies:

\[ s(t) = \cos \omega t = \frac{e^{i\omega t}}{2} + \frac{e^{-i\omega t}}{2}. \]

If we set \( \omega = 4\pi \), we could plot the spectrum of \( s(t) \) as shown in figure 2.33.

Now let’s define a complex waveform containing a single phasor:

\[ m(t) = e^{i\phi t}. \]

If we set \( \phi = -4\pi \), we could plot the spectrum of \( m(t) \) as shown in figure 2.34.

What happens if we multiply the real signal \( s(t) \) shown in figure 2.33 containing two phasors
and the complex signal \( m(t) \) shown in figure 2.34 containing just one phasor? The spectrum of the
product of the waveforms,

\[ m(t) \cdot s(t), \]

is shown in figure 2.35.

Figure 2.33
Spectrum of a real cosine signal.

Figure 2.34
Spectrum of a phasor at \(-4\pi\).
We can interpret figure 2.35 to say that all components of the real cosine signal \( s(t) \) are shifted in frequency by the frequency of \( m(t) \). In general, multiplying a signal by a phasor of frequency \( f \) adds frequency \( f \) to the frequencies of all components of the signal. All components of the signal are shifted by the same amount, no matter how complicated the signal is. (If the resulting spectrum is not conjugate symmetric, the resulting waveform is complex.)

2.10 Graphing Complex Spectra

Just as we need three dimensions to represent complex sinusoids in the time domain (as in figure 2.26), so we need three dimensions to represent complex spectra. For complex spectra, the \( z \)-axis represents frequency, and the \( y \)-axis and \( x \)-axis represent the imaginary and real number lines, respectively. The frequency of a sinusoid is represented by its position along the frequency axis (\( z \)-axis), but its magnitude is represented by a vector whose length is the sum of its real and imaginary parts.

For example, recall equation (2.52), which shows that a real cosine wave is the sum of two half-amplitude phasors with opposite frequencies. If in equation (2.52) we set \( \theta = 2\pi f \tau \), then we have:

\[
\cos 2\pi f \tau = \frac{e^{2\pi i f \tau}}{2} + \frac{e^{-2\pi i f \tau}}{2}.
\]

Figure 2.36a shows the real cosine wave in the complex time domain, and figure 2.36b shows the complex spectrum of the real cosine. Each of the bold arrows in the spectrogram corresponds to one of the phasors in the cosine equation. The position of these arrows along the frequency axis corresponds to the frequency of the positive phasor \( f \) and the frequency of the negative phasor \( -f \). The length of each arrow corresponds to the magnitude of each phasor (in this case, both have magnitudes of 1/2). The orientation around the \( x \)-axis is determined by the vector sum of the real and imaginary components of the phasors. Since the magnitudes of both phasors are real (that is, the imaginary part of their magnitudes is zero), they lie parallel to the real axis. Since the amplitudes of both phasors are positive, they are on the positive side of the real axis.

Here’s another example. If in equation (2.55) we set \( \theta = 2\pi f \tau \), then we have:

\[
\sin 2\pi f \tau = i \left( \frac{e^{-2\pi i f \tau}}{2} - \frac{e^{2\pi i f \tau}}{2} \right).
\]
Musical Signals

a) Complex Time Domain

b) Complex Frequency Domain
(Complex Spectrogram)

\[ \cos(2\pi ft) = \frac{e^{j2\pi ft}}{2} + \frac{e^{-j2\pi ft}}{2} \]

Figure 2.36
Cosine wave in the complex time and frequency domains.

\[ \sin(2\pi ft) = \frac{e^{-j2\pi ft}}{2} + \frac{e^{j2\pi ft}}{2} \]

Figure 2.37
Sine wave in the complex time and frequency domains.

This can be graphed as in figure 2.37. Both components are imaginary, so they are parallel to the imaginary axis. The positive-frequency component has a negative imaginary magnitude, so it points down.

Some useful terminology: components of a spectrum that are parallel to the real axis are said to be in phase, and those that are parallel to the imaginary axis are said to be in quadrature phase. By this definition, the cosine wave’s components are purely in phase, and the sine wave’s components are purely in quadrature phase.

2.10.1 Graphical Proof of Euler’s Formula

We can use 3-D representation of spectra to demonstrate a graphical proof of Euler’s formula, equation (2.32),

\[ e^{ix} = \cos x + i\sin x \]
First, let $z = \omega t$ so $e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$. Let’s start by graphing $\sin(\omega t)$, as shown at the top left of figure 2.38. Next, we must multiply this by $i$, which has the effect of rotating both phasors counterclockwise by $90^\circ$, as shown at the lower left of that figure. Last, we must add $\cos(\omega t)$ to the rotated phasors. Note that the negative-frequency components cancel, leaving only the positive component, $e^{j\omega t}$.

What is the complex spectrum of a real signal with a phase offset? Consider the spectrum of the real signal $\cos(\omega t + \phi)$ (figure 2.39). Adding a phase offset to a real signal rotates its components in the complex spectral domain around the frequency axis. Notice that adding a phase offset to the cosine wave rotates the positive frequency phasor counterclockwise and the negative frequency phasor clockwise.

### 2.10.2 Frequency Components of Real Signals

Here are some useful conclusions to draw from the preceding examples:

- The magnitudes of the components of real signals are always balanced between negative and positive frequencies. For example, figures 2.36 and 2.37 show the positive- and negative-frequency
components balanced in magnitude. If \(|X(f)|\) represents the magnitude of a signal at frequency \(f\), then for real signals we always have \(|X(f)| = |X(-f)|\). This is true for real sines, real cosines, and in general, real sinusoids with any phase angle.

- The positive- and negative-frequency components of real signals that are in phase (that lie along the real axis) always have even symmetry around 0 Hz, that is, their phasors point in the same direction. For example, in figure 2.36, the components are real and in phase, and have even symmetry around 0 Hz (that is, both point in the same direction). These signals are invariably real cosine signals.

- The positive- and negative-frequency components of real signals that are in quadrature phase (that lie along the imaginary axis) always have odd symmetry around 0 Hz, that is, their phasors point in opposite directions. For example, in figure 2.37, the components are imaginary and in quadrature phase, and have odd symmetry around 0 Hz (that is, they point in opposite directions). These signals are invariably real sine signals.

For real signals, these rules always obtain. Therefore, the negative- and positive-frequency components of any real signal will, when summed, always cancel any imaginary magnitude, resulting in a signal that lies entirely along the real axis. This is what equations (2.52) and (2.55) show.

Complex signals have no such restrictions. When the negative- and positive-frequency components of a complex signal are combined, the result does not necessarily cancel the imaginary part.

**If the real part of a signal is an even function, and its imaginary part is an odd function, its spectrum is said to be Hermitian.**

**The spectrum of every real signal is Hermitian.**

The symmetry of a Hermitian spectrum allows us to discard all negative-frequency spectral information of a real signal because it is redundant with the positive-frequency information. We can regenerate it later if necessary because we know by the preceding rules exactly what the negative-frequency components will be with respect to their positive-frequency counterparts. This is why spectral plots of real signals are typically displayed only for positive frequencies: we can easily infer the negative frequencies by reflecting the positive frequencies around 0 Hz. In contrast, the spectrum of a complex signal must be explicitly specified over both positive and negative frequencies because the positive-frequency and negative-frequency components of a complex signal are independent.

### 2.11 Analytic Signals

A function is said to be **analytic** if it has no negative frequencies. Analytic signals provide a convenient spectral representation of real signals because they remove the redundant negative-frequency information in such a way that it can be restored if needed.
2.11.1 Hilbert Transform

The method of creating an analytic signal from a real signal is based on the Hilbert transform. We have already seen an example of the process in figure 2.38, where a single positive-frequency phasor is created from a real cosine plus a real sine multiplied by $i$. The Hilbert transform can be used to create signals in quadrature.

The Hilbert transform of a signal $x(t)$ is another signal $y(t)$ whose frequency components are all phase-shifted by $90^\circ$ ($-\pi/2$ radians):

$$y(t) = H\{x(t)\},$$

(Hilbert Transform (2.62))

where $H\{ \}$ is the Hilbert transform.

**Using the Hilbert Transform to Create an Analytic Signal** We can interpret the processing in figure 2.38 using the Hilbert transform as follows. For some real frequency $\omega$,

$$x(t) = A \cos \omega t.$$  
Begin with a real input signal.

$$y(t) = H\{x(t)\}$$  
Apply Hilbert transform (phase shift by $90^\circ$) to create $y(t)$.

$$= A \cos \left( \omega t - \frac{\pi}{2} \right)$$  

$$= A \sin \omega t.$$  

$$z(t) = x(t) + iy(t)$$  
Multiply $y(t)$ by $i$ and combine with input signal to create analytic signal $z(t)$.

$$= A \cos \omega t + i(A \sin \omega t)$$  

$$= Ae^{j\omega t}.$$  

The result, $z(t)$, is an analytic signal because the resulting phasor $Ae^{j\omega t}$ is a single positive amplitude phasor representing a single positive frequency component that has no complementary negative-frequency component.

**Using the Hilbert Transform for Arbitrary Signals** The preceding method converts individual components of real signals to analytic form, but real-world signals tend to have many components in combination. We must generalize the procedure to all sinusoids in order to apply the Hilbert transform to more complicated signals.

Start by recalling that multiplying a phasor by $e^{j\pi/2} = i$ causes it to rotate $90^\circ$ counterclockwise, and multiplying by $e^{-j\pi/2} = -i$ causes it to rotate $90^\circ$ clockwise. We can create the quadrature signal $y(t)$ for any real signal $x(t)$ as follows:

- Rotate the positive-frequency components of $x(t)$ counterclockwise $90^\circ$ by multiplying them by $i$.
- Rotate the negative-frequency components clockwise $90^\circ$ by multiplying them by $-i$.

This procedure is shown graphically for real cosine and sine signals in figure 2.40. Notice that putting a cosine signal through the Hilbert transform produces a sine signal, and putting a sine signal through
the Hilbert transform produces a negative cosine signal. A negative cosine signal passed through the Hilbert transform will produce a negative sine signal, and one more transformation will return it to the original cosine signal. We can also observe this by delaying sinusoids successively by 90° in the time domain (figure 2.41). The Hilbert transform is sometimes called a quadrature filter because it generates these four cyclic transformations of sinusoids (see section 3.9).

2.11.2 Creating an Analytic Signal

Applying the Hilbert transform is the first step in creating an analytic signal. To complete the operation, we multiply the Hilbert transform output $y(t)$ by $i$ to rotate it an additional 90° counterclockwise. When
we add this signal to the original signal \( x(t) \), the negative frequencies cancel, leaving just the positive frequencies. This process is pictured in figure 2.42 with a cosine wave input (shown in complex form).

Summing up, the analytic signal \( x_a(t) \) of real signal \( x(t) \) is

\[
x_a(t) = x(t) + i\mathcal{H}\{x(t)\} = x(t) + i\dot{x}(t).
\]

Figure 2.43 shows the process graphically for sines and cosines, although in fact this procedure will convert any real sinusoid into an analytic signal. Note that the amplitude of the analytic signal is doubled with respect to its original real signal. (The component at 0 Hz, the DC component, remains unchanged.) That's because this process effectively wraps all negative frequencies onto their corresponding positive frequencies with a phase inversion, thereby doubling their amplitude. So this is an energy-conserving process. Because it is energy-conserving, the Hilbert-transformed signal can be regenerated into the original real input signal \( x(t) \).

### 2.11.3 Applications of the Hilbert Transform

The Hilbert transform has many musically relevant applications that are presented in later chapters. In the meantime, here are two straightforward applications.

**Envelope Follower** We can use the Hilbert transform to extract the time-varying amplitude envelope from a musical tone. Suppose the tone has a waveform \( \cos \omega t \) and an amplitude envelope \( A(t) \) so that the tone's waveform is defined as \( x(t) = A(t) \cos \omega t \). If the rate at which \( A(t) \) changes...
Creating analytic signals using the Hilbert transform.

Figure 2.43
is sufficiently slow compared to \( \omega \), then it’s reasonably safe to say that the tone’s Hilbert transform is approximately \( y(t) \equiv A(t) \sin \omega t \). Constructing the analytic signal,

\[
z(t) = x(t) + iy(t) = A(t)e^{i\omega t},
\]

and since \( |e^{i\text{anything}}| = 1 \),

\[
A(t) = |z(t)|.  \tag{2.64}
\]

Equation 2.64 says that the amplitude envelope of a signal is the absolute value of its analytic signal through time if the rate at which \( A(t) \) changes is sufficiently slow compared to \( \omega \). This is a relatively painless way of extracting the amplitude envelope of a signal though it works only for sinusoidal or quasi-sinusoidal signals. In fact, it’s really too good to be true, because though it seems that we’ve managed to extract instantaneous amplitude for all times \( t \) in \( A(t) \), in practice the best we can do is to get local amplitude, not true instantaneous amplitude. For non-sinusoidal signals, it is common to follow the Hilbert transform with a lowpass filter.

**Frequency Detector** Using the definitions for \( x(t), y(t), \) and \( z(t) \), we can express the instantaneous phase of a signal as

\[
\psi(t) = \tan^{-1} \frac{y(t)}{x(t)}.  \tag{2.65}
\]

Equation 2.65 says that the phase angle at time \( t \) equals the arctangent of the ratio of the Hilbert-transformed input signal to the input signal. The instantaneous frequency is the derivative of instantaneous phase \( \psi(t) \) with respect to time. We will learn about the derivative in chapter 6. Specifically, the instantaneous frequency is

\[
f(t) = \frac{1}{2\pi} \frac{d}{dt} \psi(t).  \tag{2.66}
\]

Once again, practical systems provide local frequency information, not true instantaneous frequency. Other interesting effects such as frequency shifting are also possible using analytic signals (see chapter 9). The Hilbert transform is of fundamental importance to many disciplines as diverse as quantum mechanics and modern music-encoding technologies such as MP3 (see chapter 10).

**Summary**

The imaginary number \( i \) was invented to allow the square of a number to be negative. Complex numbers were created so that imaginary and real numbers could coexist in the same quantity.

This in turn required understanding how complex numbers can be manipulated arithmetically. Multiplication, for example, consists of multiplying the vector lengths of two complex numbers
Musical Signals

and adding their angles. Conjugation negates the sign of the imaginary part of a complex number. Complex numbers can be graphed easily by assigning the x-axis to the real part and the y-axis to the imaginary part. Graphing numbers on the complex plane shows that multiplying a number by $i$ rotates it counterclockwise by 90° and dividing a number by $i$ rotates it clockwise 90°. Complex numbers provide a compact, powerful representation for sinusoids because we can treat complex numbers as vectors spinning on the complex plane.

Raising a complex number with unity magnitude to a power $n$ is equivalent to multiplying the angle of the complex number by $n$ (de Moivre's theorem). By combining the Taylor series for sine and cosine, and relating it to the series for $e$, we found the "most beautiful formula," $e^{i\pi} + 1 = 0$.

Adding a polar representation to complex numbers results in phasors. Adding time to a phasor makes it spin around the unit circle at a particular frequency. Positive-frequency phasors spin counterclockwise; negative-frequency phasors spin clockwise. Like the motion of a turntable, the phasor $e^{i\theta}$ embodies both the cosine and sine relations simultaneously. We observed a 90° relation between sine and cosine motion, called quadrature. Sinusoids are simply a generalization of phasor rotation, allowing us to project across the complex circle from an arbitrary position. Positive and negative frequencies result from reversing the direction of the phasor.

By investigating conjugate symmetrical phasors, we found that a real cosine is made up of the vector sum of two half-amplitude phasors of opposite frequency. Similarly, a real sine is made up of the vector difference of two imaginary half-amplitude phasors of opposite frequency.

The cosine function is called an even function, the sine function is called an odd function. With the exception of the zero function $f(x) = 0$, all functions are either even or odd, or a mixture of the two, and we found ways to break down any function into its even and odd functional components.

By injecting time into the phasor, we found that the frequency of a phasor in Hertz is the ratio of the angular velocity $\theta$ to the number of radians in a circle. Multiplying phasors together modulates their frequency, making them spin faster; demodulating them makes them spin slower.

By examining complex spectra, we found that the strengths of the frequency components of real signals are always balanced between negative and positive frequencies, have even symmetry around 0 Hz, and are in quadrature phase. If the real part of a signal is an even function, and its imaginary part is an odd function, its spectrum is said to be Hermitian. The spectrum of every real signal is Hermitian. The negative- and positive-frequency components of any real signal will, when summed, always cancel any imaginary magnitude, resulting in a signal that lies entirely along the real axis. The symmetry of a Hermitian spectrum allows us to discard all negative-frequency spectral information of a real signal because it is redundant.

A function is said to be analytic if it has no negative frequencies. Analytic signals provide a convenient spectral representation of real signals because they remove the redundant negative-frequency information in such a way that it can be restored if needed. The Hilbert transform of a signal is another signal whose frequency components are all phase-shifted by 90°. To create an analytic signal, apply the Hilbert transform, then multiply the Hilbert transform output $i$, and add this signal to the input signal. The negative frequencies cancel, leaving just the positive frequencies.