You must complete 3 of problems 1 – 4, and then problem 5 is mandatory. Each problem is worth 25 points. Please leave blank, or draw an X through, or write “Do Not Grade,” on the problem you are eliminating; I will grade the first 3 I get to if I can not figure out your intention. If answers are on the back of the page please tell me so. **Circle final answers and show all work.**

**Problem One.** Consider the following sine wave (call it \( \sigma_1 \)) generated by an expression of the form \( y = A \cdot \sin(...x.....) \) over a 0.1 second interval.

(a) Give the period (in sec), frequency (in Hz), amplitude, and phase (starting from the beginning of the interval) of \( \sigma_1 \).

(b) Suppose we take this wave and add \( \pi \) radians to its phase, effectively shifting it to the left (call this new wave \( \sigma_2 \)). Draw \( \sigma_2 \).

(c) Show the wave that results if we add \( \sigma_1 \) and \( \sigma_2 \).

(d) Show the wave that results if we multiply \( \sigma_1 \) and \( \sigma_2 \).

(e) Show the wave that results if we square \( \sigma_1 \).

For b-e you can just draw a picture, but make it precise!

Solution: (a) Period: 0.05 sec; Freq: \( 1/0.05 = 20 \text{ Hz} \); Amp: 3; Phase: \( \pi/2 \) rad (or \(-3\pi/2\))
(c) The two waves cancel out, producing all 0’s:
**Problem Two** Suppose you want to generate a pure wave for A 440 at amplitude 10,000 for exactly 1 second using a sample rate of 44100 as usual. In this question we want to explore representing this several different ways.

(a) What is the value of the angular increment $\omega$ in the expression

$$
\begin{align*}
N &= 44100 \\
A &= 10000 \\
sample &= 0.0 \\
\text{for } i \text{ in range}(N): \\
\text{sample }+=& \ A * \cos(\omega * i/N)
\end{align*}
$$

(b) Suppose we want to represent this wave as a phasor, with the (naïve) understanding that a phasor $e^{i\theta}$ (written as a complex exponential) represents a real wave $\sigma$ if $\sigma$ is just the “real part” of $e^{i\theta}$ (i.e., the projection of $e^{i\theta}$ onto the real number line). Give the complex exponential form of this phasor.

(c) But you know that the representation of this wave as in (b) is naïve, and really we need to represent this real wave as the sum of two phasors in complex conjugacy. Give the complex exponential form of these two phasors.

Solution: (a) $\omega = 2*\pi*440 = 880*\pi = 2764.52$

(b) $10000*e^{i*880*\pi/N}$

(c) $5000*e^{i*880*\pi/N} + 5000*e^{-i*880*\pi/N}$ (two half amplitude phasors of opposite frequency)
Problem Three. In this problem we assume, as usual, a sample frequency of 44100 and consider the relationship between window size and the frequencies of integral frequencies. Consider the following (integral) wave in a window of 100 samples:

(a) What is the frequency of this wave in Hz? What is the frequency in Hz of the fundamental frequency (which oscillates once in the window)?
(b) The Nyquist Limit will put an upper bound on the frequencies that can be detected with this window of 100 samples. What is the relative frequency (e.g., \( k*f \), where \( f \) is the fundamental frequency of the window) of the highest positive frequency detectable? Give this frequency in relative terms (specifying \( k \)) and also in Hz.
(c) Generalizing the previous questions, if we have a window consisting of \( N \) samples, what is the frequency in Hz of the fundamental frequency (give as a function of \( N \))?
(d) What is the highest frequency representable by a window of size \( N \)?

Solution: (a) The period of the window is 100/44100 sec and hence the fundamental frequency is 44100/100 = 441 Hz. The wave shown is 3* fundamental, so it is 1323 Hz.

(b) The Nyquist Limit is two samples per cycle, or 50*f, so the highest detectable frequency is 49*f. This is 49*441 = 21609 Hz.

(c) \( 44100/N \)

(d) It is tempting to think that the highest frequency detectable is just 1 less than the Nyquist Limit for 44100, or 22049 in all cases, but the problem is that the frequency has to be a harmonic of the window, and this depends on \( N \). The solution is to generalize (b), by figuring out the highest integral frequency under the Nyquist Limit, and multiplying by the fundamental. It turns out to matter whether \( N \) is even or odd. If \( N \) is even, we get the formula

\[
(N/2 - 1) * 44100/N = 22050 - 44100/N
\]

and if \( N \) is odd we get:

\[
(N/2 - 0.5)* 44100/N = 22050 - 22050/N
\]
**Problem Four.** In this problem you will describe various aspects of the Hann Window function.

(a) Describe the Hann Window technique, draw a picture of it, and then draw a picture of the wave in the problem 3 after the Hann Window has been applied to it.

(b) Suppose you sum the sample values in the wave shown in Problem 3—naturally, because it is an integral frequency, the samples sum to 0. However, if we apply the Hann Window, will the samples still add to 0? Answer yes or no and then discuss the reasons for your answer.

(c) What problem does the Hann Window technique attempt to address? How well does it work?

Solution: (a) The Hann Window technique applies an amplitude envelope consisting of a shifted cosine wave. Here is a picture of the effect of the Hann Window on the previous wave:

![Original Wave](image1)

![Windowed Wave](image2)

![Hann Window](image3)

(b) This is a tricky question: In the limit, this will be true (i.e., as the integral frequencies get larger and large); for smaller integral frequencies, there may be some residuals left over, e.g., consider what happens when multiplying by the fundamental frequency, which is much like the inverse of the Hann window function amplified by 6 and shifted down by 3.0. This means that the Hann window function adds some noise to the DFT algorithm, the more so as the frequencies are lower.

(c) The Hann Window technique is an attempt to eliminate the effect of non-integral frequencies in the DFT; by reducing the amplitude at each end of the window, when a frequency does not complete its period by the end of the window does not influence the outcome as much. The Hann window seems to help sharpen the spectrum of the DFT/FFT by reducing spray to some extent. Not a magic bullet, but an improvement.

**NOTE:** Not applicable this term, this is on material we will cover in the second third of the term.
Problem Five (Mandatory). Suppose you run the DFT (see next page for Python DFT) on a window of 10 real numbers representing a compound sine wave, and it produces an array of 10 complex numbers representing the amplitude and phase of 10 frequencies. The first four numbers in the raw array returned by the DFT without further processing are:

0: ( 0.0, - )  // phase for 0 amplitudes have been suppressed
1: ( 50.0, -1.57 )
2: ( 12.5, -1.57 )
3: ( 0.0, - )
4: ( 0.0, - )
.....

(a) What will be the values for the rest of the numbers (5 more)?

0: ( 0.0, - )
1: ( 50.0, -1.57 )
2: ( 12.5, -1.57 )
3: ( 0.0, - )
4: ( 0.0, - )
5: ( 0.0, - )
6: ( 0.0, - )
7: ( 0.0, - )
8: ( 12.5, 1.57 )
9: ( 50.0, 1.57 )

NOTE: Not applicable this term, this is on material we will cover in the second third of the term.
(b) Assuming we only want to discuss frequencies (positive and negative) within the Nyquist Limit, what frequency components do these numbers represent? Draw a diagram of the spectrum of this wave using all the frequencies represented by the output of the DFT. Be sure to show the amplitudes.

(c) If we want to represent only real waves (formed by two phasors in complex conjugacy) what would the spectrum be? Draw a diagram of this spectrum (don’t forget amplitudes).
(d) What happens to the raw output (listing frequencies 0 .. 9) if we change the sign of the “probe phasor” from negative to positive in the algorithm (i.e., replace \(-i\) by simply \(i\))?

Describe and explain why the output looks like this, referring to the basic operations of the DFT algorithm.

**Solution:**

The answer is that you flip the negative and positive frequencies (think of it this way: you are multiplying a wave \(w\) by a probe wave \(p\), and \(w \times -p\) is the same as \(-w \times p\)). The amplitudes are identical (half amplitude phasors in complex conjugacy), but the phases are swapped, and that is the only thing that looks different:

0: ( 0.0, - )
1: ( 50.0, 1.57 )
2: ( 12.5, 1.57 )
3: ( 0.0, - )
4: ( 0.0, - )
5: ( 0.0, - )
6: ( 0.0, - )
7: ( 0.0, - )
8: ( 12.5, -1.57 )
9: ( 50.0, -1.57 )

If you were to recover the real spectrum according to the usual method, you would get the following:

0: ( 0.0, - )
1: ( 100.0, 1.57 )
2: ( 25.0, 1.57 )
3: ( 0.0, - )
4: ( 0.0, - )

in which the phases are wrong, because this represents the following waves:
def DFT(x):
    N = len(x)
    X = [zero] * N
    mi2pidN = -i * 2.0 * pi / N
    for k in range(N):
        for n in range(N):
            xn = complex(x[n], 0.0)
            X[k] += xn * exp(mi2pidN * n * k)
        X[k] = X[k] / N
    return X