Taylor’s Theorem

1. Introduction. Suppose \( f \) is a one-variable function that has \( n + 1 \) derivatives on an interval about the point \( x = a \). Then recall from Ms. Turner’s class the single variable version of Taylor’s Theorem tells us that there is exactly one polynomial \( p \) of degree \( \leq n \) such that

\[
p(a) = f(a), \quad p'(a) = f'(a), \ldots, p^{(n)}(a) = f^{(n)}(a).
\]

This polynomial is given by

\[
p(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!} (x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!} (x - a)^n
\]

We also know the difference between \( f(x) \) and \( p(x) \):

\[
f(x) - p(x) = \frac{f^{(n+1)}(\xi)}{(n + 1)!} (x - a)^{n+1},
\]

where \( \xi \) is somewhere between \( a \) and \( x \).

The polynomial \( p \) is called the **Taylor Polynomial** of degree \( \leq n \) for \( f \) at \( a \).

Before we worry about what the Taylor polynomial might be in higher dimensions, we need to be sure we understand what is a polynomial in more than one dimension. In two dimensions, a polynomial \( p(x,y) \) of degree \( \leq n \) is a function of the form

\[
p(x,y) = \sum_{i+j \leq n} a_{ij} x^i y^j.
\]

Thus a polynomial of degree \( \leq 2 \) (perhaps more commonly known as a quadratic) looks like

\[
p(x,y) = a_{00} + a_{10} x + a_{01} y + a_{11} xy + a_{20} x^2 + a_{02} y^2.
\]

I hope it easy to guess what one means by a polynomial in three variables, \( (x,y,z) \), or indeed, in any number of variables.

Now, how might we extend the idea of the Taylor polynomial of degree \( \leq n \) for a function \( f \) at a point \( a \) ? Simple enough. It’s a polynomial \( p(x) \) of degree \( \leq n \) so that

\[
\frac{\partial^{i_1+\ldots+i_q} f(a)}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_q^{i_q}} = \frac{\partial^{i_1+\ldots+i_q} p(a)}{\partial x_1^{i_1} \partial x_2^{i_2} \ldots \partial x_q^{i_q}},
\]

for all \( i_1, i_2, \ldots, i_q \) such that \( i_1 + i_2 + \ldots + i_q \leq n \).

This looks pretty ferocious in general, so let’s see what it says for just two variables. In this case, we have \( a = (a,b) \) and the Taylor polynomial \( p(x,y) \) at \( a \) becomes the polynomial such that
\[
\frac{\partial^{ij} f(a)}{\partial^i x \partial^j y} = \frac{\partial^{ij} p(a)}{\partial^i x \partial^j y},
\]
for all \(i + j \leq n\).

Example

Let \(f(x, y) = \cos(x + y)\), and let \(p(x, y) = 1 - \frac{x^2}{2} - xy - \frac{y^2}{2}\). Let’s verify that \(p\) is the Taylor polynomial of degree \(\leq 2\) for \(f\) at \((0, 0)\). He we go.

\[
\begin{align*}
f(0, 0) &= 1, \text{ and } p(0, 0) = 1; \\
\frac{\partial f}{\partial x} &= -\sin(x + y), \text{ and } \frac{\partial p}{\partial x} = -x - y; \\
\frac{\partial f}{\partial y} &= -\sin(x + y), \text{ and } \frac{\partial p}{\partial y} = -x - y; \\
\frac{\partial^2 f}{\partial x^2} &= -\cos(x + y), \text{ and } \frac{\partial^2 p}{\partial x^2} = -1, \\
\frac{\partial^2 f}{\partial y^2} &= -\cos(x + y), \text{ and } \frac{\partial^2 p}{\partial y^2} = -1, \\
\frac{\partial^2 f}{\partial x \partial y} &= -\cos(x + y), \text{ and } \frac{\partial^2 p}{\partial x \partial y} = -1.
\end{align*}
\]

Now it’s easy to see that

\[
\begin{align*}
f(0, 0) &= 0 = p(0, 0); \\
\frac{\partial f}{\partial x} (0, 0) &= 0 = \frac{\partial p}{\partial x} (0, 0); \\
\frac{\partial f}{\partial y} (0, 0) &= 0 = \frac{\partial p}{\partial y} (0, 0); \\
\frac{\partial^2 f}{\partial x^2} (0, 0) &= -1 = \frac{\partial^2 p}{\partial x^2} (0, 0); \\
\frac{\partial^2 f}{\partial y^2} (0, 0) &= -1 = \frac{\partial^2 p}{\partial y^2} (0, 0); \text{ and} \\
\frac{\partial^2 f}{\partial x \partial y} (0, 0) &= -1 = \frac{\partial^2 p}{\partial x \partial y} (0, 0).
\end{align*}
\]

Exercises

1. Verify that the polynomial in the Example is also the Taylor polynomial for \(f\) at \((0, 0)\) of degree \(\leq 3\).

2. Let \(f(x, y) = \sin(x + y)\). Which of the following is the Taylor polynomial of degree \(\leq 2\) for \(f\) at \((0, 0)\)? Explain.
   a) \(p(x, y) = 1 + x^2 + y^2\)  
   b) \(p(x, y) = xy\)
c) \( p(x, y) = x^2 + xy + 2y \)

\[ d) p(x, y) = x + y \]

2. **Derivatives.** Prior to finding a general recipe for the Taylor polynomial, we need look at finding higher order derivatives of certain composite functions. Let \( f \) be a real-valued function defined on a subset of \( \mathbb{R}^q \). Suppose that in a neighborhood of the point \( x \), the function \( f \) has a lot of continuous partial derivatives. Define the function \( g \) by

\[ g(t) = f(a + th), \]

where \( a = (a_1, a_2, \ldots, a_q) \) and \( h = (h_1, h_2, \ldots, h_q) \). We know from the chain rule that \( g'(t) \) is given by

\[ g'(t) = \nabla f(a + th) \cdot h \]

\[ = \left( \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_q} \right) \cdot (h_1, h_2, \ldots, h_q) \]

\[ = \left( h_1 \frac{\partial f}{\partial x_1} + h_2 \frac{\partial f}{\partial x_2} + \ldots + h_q \frac{\partial f}{\partial x_q} \right) f \bigg|_{(a+th)} \]

In keeping with our general practice of restricting ourselves to dimensions one, two, or three, let’s look first at the case \( q = 2 \). As usual, we’ll write \( x = (x, y) \) and \( h = (h, k) \). The expression for \( g'(t) \) now looks like:

\[ g'(t) = \left( h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right) f \bigg|_{(x+th)} \]

We are now in business, for we have a nice recipe for higher order derivatives of \( g \):

\[ g^{(m)}(t) = \left( h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right)^m f \bigg|_{(x+th)} \]

For example,

\[ g''(t) = \left( h \frac{\partial f}{\partial x} + k \frac{\partial f}{\partial y} \right)^2 f \]

\[ = \left( h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \right) f \]

\[ = h^2 \frac{\partial^2 f}{\partial x^2} + 2hk \frac{\partial^2 f}{\partial x \partial y} + k^2 \frac{\partial^2 f}{\partial y^2} \]

**Example**

Suppose \( f(x, y) = x^2y^3 + y^2 \). Let’s find the second derivative of the function

\[ g(t) = f(1 + 3t, -2 + t) \]
First,
\[ g''(t) = \left(3 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^2 f \]
\[ = 9 \frac{\partial^2 f}{\partial x^2} + 6 \frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \]
Now, \( \frac{\partial f}{\partial x} = 2xy^3 \), and \( \frac{\partial f}{\partial y} = 3x^2y^2 + 2y \), and so \( \frac{\partial^2 f}{\partial x \partial y} = 2y^3 \), \( \frac{\partial^2 f}{\partial y^2} = 6y^2 \), and \( \frac{\partial^2 f}{\partial x^2} = 6x^2y + 2 \). Thus,
\[ g''(t) = 18(-2 + t)^3 + 36(-2 + t)^2 + 6(1 + 3t)^2(-2 + t) + 2 \]

**Exercises**

3. Let \( f(x, y) = xe^y \). Find the derivative of \( g(t) = f(1 + t, 3 - 4t) \).

4. Find the second derivative of the function \( g \) defined in **Problem 3**.

5. Let \( F(u, v) = u^3v + v^2 \). Find the second derivative of \( R(z) = F(z, 3z) \).

6. Find \( g'''(t) \), where \( g \) is the function defined in the Example.

3. **The Taylor polynomial.** To find the Taylor polynomial for a function \( f \) of several variables at a point \( a \), we shall simply apply the one-dimensional results to the function \( g(t) = f(a + th) \).

Thus,
\[ g(t) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} t^m + \frac{g^{(n+1)}(\xi)}{(n + 1)!} t^{n+1}, \]
where \( \xi \) is a number between 0 and \( t \). Next, substitute \( t = 1 \) into the above:
\[ g(1) = f(a) = \sum_{m=0}^{n} \frac{g^{(m)}(0)}{m!} + \frac{g^{(n+1)}(\xi)}{(n + 1)!} \]

We know the value of \( g^{(k)} \) from **Section 2**:
\[ f(a + h) = \sum_{m=0}^{n} \frac{1}{m!} \left( h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \ldots + h_q \frac{\partial}{\partial x_q} \right)^m f(a) \]
The point \( c \) lies somewhere on the line segment joining \( a \) and \( a + h \).

The polynomial

\[
p(h) = p(h_1, h_2, \ldots, h_q) = \sum_{m=0}^{n} \frac{1}{m!} \left( h_1 \frac{\partial}{\partial x_1} + h_2 \frac{\partial}{\partial x_2} + \ldots + h_q \frac{\partial}{\partial x_q} \right)^m f(a)
\]

is the Taylor polynomial of degree \( \leq n \) for \( f \) at \( a \); the last term is traditionally called the error term or sometimes, the remainder term. Actually, if we let \( h = x - a \), then \( q(x) = p(x - a) \) is the thing we called the Taylor polynomial in the first section.

This is pretty fierce looking. Let’s look at the two variable case:

\[
f(a_1 + h, a_2 + k) = \sum_{m=0}^{n} \frac{1}{m!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^m f(a_1, a_2) + \frac{1}{(n+1)!} \left( h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^{n+1} f(c_1, c_2)
\]

where \( (c_1, c_2) \) is on the line joining \( (a_1, a_2) \) and \( (a_1 + h, a_2 + k) \).

**Example**

Let \( f(x, y) = \sin x \sin y \). For \( n = 2 \) and \( a = (0, 0) \), Taylor’s polynomial becomes

\[
p(h, k) = f(0,0) + h \frac{\partial f}{\partial x}(0,0) + k \frac{\partial f}{\partial y}(0,0) + \frac{h^2}{2} \frac{\partial^2 f}{\partial x^2}(0,0) + \frac{hk}{2} \frac{\partial^2 f}{\partial x \partial y}(0,0) + \frac{k^2}{2} \frac{\partial^2 f}{\partial y^2}(0,0)
\]

We have

\[
\frac{\partial f}{\partial x} = \cos x \sin y; \quad \frac{\partial f}{\partial y} = \sin x \cos y; \quad \frac{\partial^2 f}{\partial x^2} = -\sin x \sin y; \quad \frac{\partial^2 f}{\partial x \partial y} = \cos x \cos y; \quad \frac{\partial^2 f}{\partial y^2} = -\sin x \sin y.
\]

Thus,

\[
p(h, k) = hk.
\]

Let’s get an estimate for how well this approximates \( \sin x \sin y \) near \( (0,0) \). We know that

\[
|\sin x \sin y - xy| = \left| \frac{1}{3!} \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^3 f(\xi, \mu) \right|
\]

where \((\xi, \mu)\) is one the segment joining \((x,y)\) and the origin. Now,
\[
\left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^3 f = x^3 \frac{\partial^3 f}{\partial x^3} + 3x^2 y \frac{\partial^3 f}{\partial x^2 \partial y} + 3xy^2 \frac{\partial^3 f}{\partial x \partial y^2} + y^3 \frac{\partial^3 f}{\partial y^3}.
\]

Next, let’s suppose that \(|x| \leq c\) and \(|y| \leq c\) for some constant \(c\). Noting that all the partial derivatives in the above expression are simply products of sine and cosines, we can estimate

\[
\left| \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right)^3 f \right| \leq 8c^3,
\]

and so, at last,

\[
|\sin x \sin y - xy| \leq \frac{8c^3}{6} = \frac{4}{3} c^3
\]

**Exercises**

7. Find the Taylor polynomial of degree \(\leq 1\) for \(f(x, y) = e^{xy}\) at \((0, 0)\).

8. Find the Taylor polynomial of degree \(\leq 2\) for \(f(x, y) = e^{xy}\) at \((0, 0)\).

9. Find the Taylor polynomial of degree \(\leq 3\) for \(f(x, y) = e^{xy}\) at \((0, 0)\).

10. Find the Taylor polynomial of degree \(\leq 1\) for \(f(x, y) = e^x \cos y\) at \((0, 0)\).

11. Use Taylor’s Theorem to find a quadratic approximation of \(e^x \cos y\) at the origin.

12. Estimate the error in the approximation found in Problem 11 if \(|x| \leq 0.1\) and \(|y| \leq 0.1\).