Algebraic algorithms

Freely using the textbook: Victor Shoup's "A Computational Introduction to Number Theory and Algebra"

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The class structure

See the course homepage.

Mathematical preliminaries
Logic

Logical operations: $\wedge$, $\neg$, $\vee$, $\Rightarrow$, $\Leftrightarrow$. $\forall$, $\exists$.

Example

x divides y, or y is divisible by x: $x|y \Leftrightarrow \exists z(x * z = y)$.

Sets

Notation: $\{2, 3, 5\}$. $x \in A$. The empty set.
Some important sets: $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

Example

x divides y more precisely: $x|y \Leftrightarrow \exists z \in \mathbb{Z}(x * z = y)$.

Set notation using conditions:

$$\{\, x \in \mathbb{Z} : 3|x \,\} = \{\, 3x : x \in \mathbb{Z} \,\}.$$

Note that x has a different role on the left-hand side and on the right-hand side. The x in this notation is a bound variable: its meaning is unrelated to everything outside the braces.

Example

Composite numbers: $\{\, xy : x, y \in \mathbb{Z} \setminus \{-1, 1\} \,\}$.

$A \subseteq B$, $A \subset B$ will mean the same! Proper subset: $A \subsetneq B$.
Set operations: $A \cup B$, $A \cap B$, $A \setminus B$. Disjoint sets: $A \cap B = \emptyset$.
The set of all subsets of a set A is denoted by 2^A.

Functions

The notation $f : A \to B$.

Example

$g(x) = 1/(x^2 - 1)$. It maps from $\mathbb{R} \setminus \{-1, 1\}$, to $\mathbb{R}$, so

$$g : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R}. \tag{1}$$

$\text{Domain}(g) = \mathbb{R} \setminus \{-1, 1\}$.

In general,

$$\text{Range}(f) = \{ f(x) : x \in \text{Domain}(f) \}.$$

In the example,

$$\text{Range}(g) = (-\infty, -1] \cup (0, \infty) = \mathbb{R} \setminus (-1, 0].$$

Note that $(0, \infty)$ is an open interval.

We could write $g : \mathbb{R} \setminus \{-1, 1\} \to \mathbb{R} \setminus (-1, 0)$, but (1) is correct, too: it says that g is a function mapping from $\mathbb{R} \setminus \{-1, 1\}$ into $\mathbb{R}$. On the other hand, g is mapping onto $\mathbb{R} \setminus (-1, 0)$. An “onto” function is also called surjective.

Injective and surjective

A function is one-to-one (injective) if $f(x) = f(y)$ implies $x = y$.

Theorem

If a set A is finite then a function $f : A \to A$ is onto if and only if it is one-to-one.

The proof is left for exercise.
The theorem is false for infinite A.

Example

A one-to-one function that is not onto: the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 2x$.
An onto function that is not one-to-one: exercise.

We will also use the notation

$$x \mapsto 2x$$

to denote this function. (The $\mapsto$ notation is similar to the lambda notation used in the logic of programming languages.)
A function is called invertible if it is onto and one-to-one. For an invertible function $f : A \to B$, the inverse function $f^{-1} : B \to A$ is always defined uniquely: $f_{-1}(b) = a$ if and only if $f(a) = b$.
An invertible function $f : A \to A$ is also called a permutation.

Tuples

Ordered pair (x, y), unordered pair $\{x, y\}$. (The (x, y) notation conflicts with the same notation for open intervals. So, sometimes $\langle x, y\rangle$ is used.) The Cartesian product

$$A \times B = \{\, (x, y) : x \in A, y \in B \,\}.$$

A function of two arguments: we will use the notation

$$f : A \times B \to C$$

when $f(x, y) \in C$ for $x \in A$, $y \in B$. Indeed, f can be regarded as a one-argument function of the ordered pair (x, y).
Ordered triple, and so on. Sequence $(x_1, \dots, x_n)$.

Inverse image

For a function $f : A \to B$, and a set $C \subseteq A$ we will write

$$f(C) = \{ f(x) : x \in A \}.$$

Thus, $\text{Range}(f) = f(A)$.
Example: $2\mathbb{Z}$ is the set of even numbers.
For $D \in B$, we will write

$$f^{-1}(D) = \{ x : f(x) \in D \}.$$

Note that this makes sense even if the function is not invertible.
However, $f^{-1}(D)$ is always a set, and it may be empty.

Example

If $f : \mathbb{Z} \to \mathbb{Z}$ is the function with $f(x) = 2\lfloor x/2 \rfloor$ then $f^{-1}(0) = \{0, 1\}$, $f^{-1}(\{1\}) = \emptyset = \{\}$, $f^{-1}(2) = \{2, 3\}$, $f^{-1}(\{3\}) = \emptyset$, and so on.

Partitions

A partition of a set A is a finite sequence $(A_1, \dots, A_n)$ of pairwise disjoint subsets of A such that $A_1 \cup \cdots \cup A_n = A$. Given any function $f : A \to \{1, \dots, n\}$, it gives rise to a partition $(f^{-1}(\{1\}), \dots, f^{-1}(\{n\}))$. And every partition defines such a function.

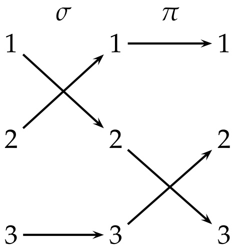
We will also talk about infinite partitions. A partition in this case is a function $p : B \to 2^A$ such that $\bigcup_{b \in B} p(b) = A$ and for $b \neq c$ we have $p(b) \cap p(c) = \emptyset$.

Operations

Functions are sometimes are also called operations. Especially, functions of the form $f : A \to A$ or $g : A \times A \to A$. For example, $(x, y) \mapsto x + y$ for $x, y \in \mathbb{R}$ is the addition operation.

Associativity. Example: functions $f : A \to A$, with the compositon operation.

Commutativity. Same example, say the permutations σ, π over $\{1, 2, 3\}$ on the right do not commute.



Distributivity. Examples: $*$ through $+$, further $\cap$ though $\cup$ and $\cup$ through $\cap$.

Relations

A binary relation is a set $R \subseteq A \times B$. We will write $(x, y) \in R$ also as $R(x, y)$ (with Boolean value). Thus

$$R(x, y) \Leftrightarrow (x, y) \in R.$$

Frequently, infix notation. Example: $x < y$, where $< \subset \mathbb{R} \times \mathbb{R}$.
Ternary relation: $R \in A \times B \times C$.
Interesting properties of binary relations over a set A.
Reflexive.
Symmetric.
Transitive.
A binary relation can be represented by a graph. If the relation is symmetric the graph can be undirected, otherwise it must be directed. In all cases, at most one edge can be between nodes.

Equivalence relation

Equivalence relation over a set A: reflexive, symmetric transitive.
Example: equality. Other example: reachability in a graph.

Theorem

A relation $R \subset A \times A$ is an equivalence relation if and only if there is a function $f : A \to B$ such that $R(x, y) \Leftrightarrow f(x) = f(y)$.

Proof: exercise.
Each set of the form $C_x = \{ y : R(x, y) \}$ is called an equivalence class.
An equivalence relation partitions the underlying set into the equivalence classes.
In a partition into equivalence classes, we frequently pick a representative in each class. Example: rays and unit vectors.

Preorder, partial order

A relation $\leqslant$ is antisymmetric if $a \leqslant b$ and $b \leqslant a$ implies $a = b$.
Preorder $\leqslant$: reflexive, transitive.
A preorder is a partial order if it is antisymmetric. Simplest example: $\leqslant$ among real numbers.

Example

The relation $\subseteq$ among subsets of a set A is a partial order.

In a preorder, we can introduce a relation $\sim$: $x \sim y$ if $x \leqslant y$ and $y \leqslant x$. This is an equivalence relation, and the relation induced by $\leqslant$ on the equivalence classes is a partial order.

Example

The relation $x|y$ over the set $\mathbb{Z}$ of integers is a preorder. For every integer x, its equivalence class is $\{x, -x\}$.

Asymptotic analysis

$O(), o(), \Omega(), \Theta()$. More notation: $f(n) \ll g(n)$ for $f(n) = o(g(n))$, $f(n) \overset{*}{<} g(n)$ for $f(n) = O(g(n))$ and $\overset{*}{=}$ for ($\overset{*}{<}$ and $\overset{*}{>}$).
The relation $\overset{*}{<}$ is a preorder. On the equivalence classes of $\overset{*}{=}$ it turns into a partial order.
The most important function classes: log, logpower, linear, power, exponential. These are not all equivalence classes under $\overset{*}{=}$.

Some simplification rules

- Addition: take the maximum. Do this always to simplify expressions. *Warning*: do it only if the number of terms is constant!
- An expression $f(n)^{g(n)}$ is generally worth rewriting as $2^{g(n)\log f(n)}$. For example, $n^{\log n} = 2^{(\log n)\cdot(\log n)} = 2^{\log^2 n}$.
- But sometimes we make the reverse transformation:

$$3^{\log n} = 2^{(\log n)\cdot(\log 3)} = (2^{\log n})^{\log 3} = n^{\log 3}.$$

The last form is easiest to understand, showing n to a constant power $\log 3$.

Examples

$$n/\log\log n + \log^2 n \stackrel{*}{=} n/\log\log n.$$

Indeed, $\log\log n \ll \log n \ll n^{1/2}$, hence $n/\log\log n \gg n^{1/2} \gg \log^2 n$.

Order the following functions by growth rate:

$$\begin{aligned}
& n^2 - 3\log\log n && \stackrel{*}{=} n^2, \\
& \log n/n, && \\
& \log\log n, && \\
& n\log^2 n, && \\
& 3 + 1/n && \stackrel{*}{=} 1, \\
& \sqrt{(5n)}/2^n, && \\
& (1.2)^{n-1} + \sqrt{n} + \log n && \stackrel{*}{=} (1.2)^n.
\end{aligned}$$

Solution:

$$\begin{aligned}
\sqrt{(5n)}/2^n \ll \log n/n \ll 1 \ll \log\log n \\
\ll n/\log\log n \ll n\log^2 n \ll n^2 \ll (1.2)^n.
\end{aligned}$$

Sums: the art of simplification

Arithmetic series.
Geometric series: its rate of growth is equal to the rate of growth of its largest term.

Example

$$\log n! = \log 2 + \log 3 + \cdots + \log n = \Theta(n \log n).$$

Indeed, upper bound: $\log n! < n \log n$.
Lower bound:

$$\begin{aligned} \log n! &> \log(n/2) + \log(n/2+1) + \cdots + \log n > (n/2)\log(n/2) \\ &= (n/2)(\log n - 1) = (1/2)n \log n - n/2. \end{aligned}$$

Examples

Prove the following, via rough estimates:

- $1 + 2^3 + 3^3 + \cdots + n^3 = \Theta(n^4)$.
- $1/3 + 2/3^2 + 3/3^3 + 4/3^4 + \cdots < \infty$.

Example

$$1 + 1/2 + 1/3 + \cdots + 1/n = \Theta(\log n).$$

Indeed, for $n = 2^{k-1}$, upper bound:

$$1 + 1/2 + 1/2 + 1/4 + 1/4 + 1/4 + 1/4 + 1/8 + \dots$$
$$= 1 + 1 + \cdots + 1 \text{ (} k \text{ times)}.$$

Lower bound:

$$1/2 + 1/4 + 1/4 + 1/8 + 1/8 + 1/8 + 1/8 + 1/16 + \dots$$
$$= 1/2 + 1/2 + \cdots + 1/2 \text{ (} k \text{ times)}.$$

Random access machine

Fixed number K of registers R_j, $j = 1, \dots, K$. Memory: one-way infinite tape: cell i contains natural number $T[i]$ of arbitrary size.

Program: a sequence of instructions, in the "program store": a (potentially) infinite sequence of registers containing instructions. A program counter.

`read j`	$R_0 = T[R_j]$	(this is random access)
`write j`		
`store j`	$R_j = R_0$	
`load j`		
`add j`	$R_0 \mathrel{+}= R_j$	
`add =c`	$R_0 \mathrel{+}= c$	
`sub j`	$R_0 = \lvert R_0 - R_j \rvert^+$	
`sub =c`		
`half`	$R_0 \mathrel{/}= 2$	
`jump s`		
`jpos s`	if $R_0 > 0$ then jump s	
`jzero s`		
`halt`		

In our applications, we will impose some bound k on the number of cells.

The size of the numbers stored in each cell will be bounded by k^c for some constant c. Thus, the wordsize of the machine will be logarithmic in the size of the memory, allowing to store the address of any position in a cell.

Basic integer arithmetic

Length of numbers

$$\operatorname{len}(n) = \begin{cases} \lfloor \log |n| \rfloor + 1 & \text{if } n \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

This is essentially the same as $\log n$, but is always defined. We will generally use $\operatorname{len}(n)$ in expressing complexities.

Upper bounds

On the complexity of addition, multiplication, division (with remainder), via the algorithms learned at school.

Theorem

The complexity of computing $(a, b) \mapsto (q, r)$ *in the division with remainder* $a = qb + r$ *is* $O(\text{len}(q)\text{len}(b))$.

Proof.

The long division algorithm has $\leqslant \text{len}(q)$ iterations, with numbers of length $\leqslant \text{len}(b)$. □

Theorem (Fundamental theorem of arithmetic)

Unique prime decomposition $\pm p_1^{e_1} \dots p_k^{e_k}$.

The proof is not trivial, we will lead up to it. We will see analogous situations later in which the theorem does not hold.

Example

Irreducible family: one or two adult and some minors.
Later: the ring $\mathbb{Z}[\sqrt{-5}]$.

The above theorem is equivalent to the following lemma:

Lemma (Fundamental)

If p *is prime and* $a, b \in \mathbb{Z}$ *then* $p|ab$ *if and only if* $p|a$ *or* $p|b$.

In class, we have shown the equivalence.

Ideals

If I, J are ideals so is $aI + bJ$.
$a\mathbb{Z} \subseteq b\mathbb{Z}$ if and only if $b|a$.
Careful: generally $a\mathbb{Z} + b\mathbb{Z} \neq (a+b)\mathbb{Z}$.

Example

$2\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$.

Principal ideal

The following theorem is the crucial step in the proof of the Fundamental Theorem.

Theorem

In $\mathbb{Z}$, every ideal I is principal.

Proof.

Let d be the smallest positive integer in I. The proof shows $I = d\mathbb{Z}$, using division with remainder. □

Corollary

If $d > 0$ and $a\mathbb{Z} + b\mathbb{Z} = d\mathbb{Z}$ then $d = \gcd(a, b)$. In particular, we found that

(a) *Every other divisor of a, b divides $\gcd(a, b)$.*

(b) *For all a, b there are $s, t \in \mathbb{Z}$ with $\gcd(a, b) = sa + tb$.*

The proof of the theorem is non-algorithmic. It does not give us a method to calculate $\gcd(a, b)$: in particular, it does not give us the s, t in the above corollary. We will return to this.

Theorem

For a, b, c *with* $\gcd a, c = 1$ *and* $c|ab$ *we have* $c|b$.

This theorem implies the Fundamental Lemma announced above.

Proof.

Using $1 = sc + ta$, hence $b = scb + tab$. □

Some consequences of unique factorization

There are infinitely many primes.
The notation $\nu_p(a)$. gcd and minimum, lcm and maximum.

$$\operatorname{lcm}(a, b) \cdot \gcd(a, b) = |ab|$$

Pairwise relatively prime numbers.
Representing fractions in lowest terms.
Lowest common denominator.

Rings

Unless stated otherwise, commutative, with a unit element. The detailed properties of rings will be deduced later (see Section 9 of Shoup, in particular Theorem 9.2). We use rings here only as examples.

Examples

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.
- The set of (say, 2×2) matrices over $\mathbb{R}$ is also a ring, but is not commutative.
- The set $2\mathbb{Z}$ is also a ring, but has no unit element.
- If R is a commutative ring, then $R[x, y]$, the set of polynomials in x, y with coefficients in R, is also a ring.

Theorem

Let R be a ring. Then:

(i) *the multiplicative identity is unique.*

(ii) $0 \cdot a = 0$ *for all* a *in* R.

(iii) $(-a)b = a(-b) = -(ab)$ *for all* $a, b \in R$.

(iv) $(-a)(-b) = ab$ *for all* $a, b \in R$.

(v) $(na)b = a(nb) = n(ab)$ *for all* $n \in \mathbb{Z}$, $a, b \in R$.

Ideals.

Example

A non-principal ideal: $x\mathbb{Z}[x,y] + y\mathbb{Z}[x,y]$ in $\mathbb{Z}[x,y]$.

Example

Non-unique irreducible factorization in a ring. Let the ring be $\mathbb{Z}[\sqrt{-5}]$.

$$6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}).$$

How to show that $2, 3, (1 + \sqrt{-5}), (1 - \sqrt{-5})$ are irreducible? Let $N(a + b\sqrt{-5}) = a^2 + 5b^2$, then it is easy to see that $N(xy) = N(x)N(y)$, since $N(z)$ is the square absolute value of the complex number z. It is always integer here.
If $N(z) = 1$ then $z = \pm 1$.
If $N(z) > 1$ then $N(z) \geqslant 4$.
For $z = 2, 3, (1 + \sqrt{-5}), (1 - \sqrt{-5})$, we have $N(z) = 4, 9, 6, 6$. The only nontrivial factors of these numbers are 2 and 3, but there is no z with $N(z) \in \{2, 3\}$.

The basic Euclidean algorithm

Assume $a \geqslant b \geqslant 0$ are integers.

$$\begin{aligned} a = r_0, \quad b &= r_1, \\ r_{i-1} &= r_i q_i + r_{i+1} \quad (0 < r_{i+1} < r_i), \quad (1 \leqslant i < \ell) \\ &\vdots \\ r_{\ell-1} &= q_\ell r_\ell \end{aligned}$$

Upper bound on the number ℓ of iterations:

$$\ell \leqslant \log_\phi b + 1,$$

where $\phi = (1 + \sqrt{5})/2 \approx 1.62$. We only note $\ell = O(\log b)$ which is obvious from

$$r_{i+1} \leqslant r_{i-1}/2.$$

Theorem

Euclid's algorithm runs in time $O(\mathrm{len}(a)\mathrm{len}(b))$.

This is stronger than the upper bound seen above.

Proof.

We have

$$\mathrm{len}(b)\sum_{i=1}^{\ell}\mathrm{len}(q_i) \leqslant \mathrm{len}(b)\sum_{i=1}^{\ell}(1+\log(q_i)) \leqslant \mathrm{len}(b)(\ell+\log(\prod_{i=1}^{\ell} q_i)).$$

Now,

$$a = r_0 \geqslant r_1 q_1 \geqslant r_2 q_2 q_1 \geqslant \cdots \geqslant r_\ell q_\ell \cdots q_1.$$

□

The extended Euclidean algorithm

$$s_0 = 1, \qquad t_0 = 0,$$
$$s_1 = 0, \qquad t_1 = 1,$$
$$s_{i+1} = s_{i-1} - s_i q_i, \qquad \text{same for } t_i.$$

Theorem

The following relations hold.

(i) $s_i a + t_i b = r_i$.

(ii) $s_i t_{i+1} - t_i s_{i+1} = (-1)^i$.

(iii) $\gcd(s_i, t_i) = 1$.

(iv) $t_i t_{i+1} \leqslant 0$, $|t_i| \leqslant |t_{i+1}|$, *same for* s_i.

(v) $r_{i-1}|t_i| \leqslant a$, $r_{i-1}|s_i| \leqslant b$.

Proof.

(i),(ii): induction. (i) follows from (ii). (iv): induction. (v): combining (i) for i and $i - 1$. □

Matrix representation

$$\begin{pmatrix} r_i \\ r_{i+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix} = Q_i \begin{pmatrix} r_{i-1} \\ r_i \end{pmatrix}.$$

Define $M_i = Q_i \cdots Q_1$, then

$$M_i = \begin{pmatrix} s_i & t_i \\ s_{i+1} & t_{i+1} \end{pmatrix}.$$

Now the relation $s_i t_{i+1} - t_i s_{i+1} = (-1)^i$ above says $\det M_i = \prod_{j=1}^{i} \det Q_i = (-1)^i$.

Congruences

$a \equiv b \pmod{m}$ if $m | b - a$.
More generally, in a ring with some ideal I, we write $a \equiv b \pmod{I}$ if $(b - a) \in I$.

Theorem

The relation $\equiv$ has the following properties, when I is fixed.

(a) *It is an equivalence relation.*

(b) *Addition and multiplication of congruences.*

Example (From Emil Kiss)

Is the equation $x^2 + 5y = 1002$ solvable among integers?
This seems hard until we take the remainders modulo 5, then it says: $x^2 \equiv 2 \pmod{5}$. The squares modulo 5 are $0, 1, 4, 4, 1$, so 2 is not a square.

The ring of congruence classes

For an integer x, let

$$[x]_m = \{ y \in \mathbb{Z} : y \equiv x \pmod{m} \}$$

denote the residue class of x modulo m. We choose a representative for each class $[x]_m$: its smallest nonnegative element.

Example

The set $[-3]_5$ is $\{\dots, -8, -3, 2, 7, \dots\}$. Its representative is 2.

Definition of the operations $+, \cdot$ on these classes. This is possible due to the additivity and multiplicativity of $\equiv$.
The set of classes with these operations is turned into a ring which we denote by $\mathbb{Z}_m$. We frequently write $\mathbb{Z}_m = \{0, 1, \dots, (m-1)\}$, that is we use the representative of class $[i]_m$ to denote the class.

Division of congruences

Does $c \cdot a \equiv c \cdot b \pmod{m}$ imply $a \equiv b \pmod{m}$ when $c \not\equiv 0 \pmod{m}$? Not always.

Example

$2 \cdot 3 = 6 \equiv 0 \equiv 2 \cdot 0 \pmod{6}$, but $3 \not\equiv 0 \pmod{6}$.

The numbers 2,3 are called here zero divisors. In general, an element $x \neq 0$ of a ring R is a zero divisor if there is an element $y \neq 0$ in R with $x \cdot y = 0$.

Theorem

In a finite ring R, if b is not a zero divisor then the equation $x \cdot b = c$ has a unique solution for each c: that is, we can divide by b.

Proof.

The mapping $x \to x \cdot b$ is one-to-one. Indeed, if it is not then there would be different elemnts x, y with $x \cdot b = y \cdot b$, but $(x - y) \cdot b \neq 0$, since b is not a zero divisor.
At the beginning of class, we have seen that in a finite set, if a class is one-to-one then it is also onto. Therefore for each c there is an x with $x \cdot b = c$. The one-to-one property implies that x is unique. □

Observe that this proof is non-constructive: it does not help finding x from b, c.
Actually we only need to find b^{-1}, that is the solution of $x \cdot b = 1$

Finding the inverse

Proposition

An element of $b \in \mathbb{Z}_m$ is not a zero divisor if and only if $\gcd(b, m) = 1$.

To find the inverse x of b, we need to solve the equation $x \cdot b + y \cdot m = 1$. Euclid's algorithm gives us these x, y, and then $x \equiv b^{-1} \pmod{m}$.

Example

Inverse of 8 modulo 15.

Characterizing the set of all solutions of the equation

$$a \cdot x \equiv b \pmod{m}.$$

Corollary (Cancellation law of congruences)

If $\gcd(c, m) = 1$ *and* $ac \equiv bc \pmod{m}$ *then* $a \equiv b \pmod{m}$.

Examples

- We have $5 \cdot 2 \equiv 5 \cdot (-4) \pmod{6}$. This implies $2 \equiv -4 \pmod{6}$.
- We have $3 \cdot 5 \equiv 3 \cdot 3 \pmod{6}$, but $5 \not\equiv 3 \pmod{6}$.

What can we do in the second case? Simplify as follows.

Proposition

For all a, b, c *the relation* $ac \equiv bc \pmod{mc}$ *implies* $a \equiv b \pmod{m}$.

The proof is immediate.
In the above example, from $3 \cdot 5 \equiv 3 \cdot 3 \pmod{6}$ we can imply $5 \equiv 1 \pmod{2}$.

Chinese remainder theorem

Consider two diferent moduli: m_1 and m_2. Do all residue classes of m_1 intersect with all residue classes of m_2? That is, given a_1, a_2, we are looking for an x with

$$x \equiv a_1 \pmod{m_1}, \quad x \equiv a_2 \pmod{m_2}.$$

There is not always a solution. For example, there is no x with

$$x \equiv 0 \pmod{2}, \quad x \equiv 1 \pmod{4}.$$

But if m_1, m_2 are coprime, there is always a solution. More generally:

Theorem

If $m_1, \ldots, m_k$ are relatively prime with $M = m_1 \cdots m_k$ then for all $a_1, \ldots, a_k \in \mathbb{Z}$ there is a unique $0 \leqslant x < M$ with $x \equiv a_i \pmod{m_i}$ for all $i = 1, \ldots, k$.

Proof.

Let $I(n) = \{0, \ldots, n-1\}$. The sets $U = I(M)$ and $V = I(m_1) \times \cdots \times I(m_k)$ both have size M. We define a mapping $f : U \to V$ as follows:

$$f(x) = (x \bmod m_1, \ldots, x \bmod m_k).$$

Let us show that this mapping is one-to-one. Indeed, if $f(x) = f(y)$ for some $x \leqslant y$ then $x \equiv y \pmod{m_i}$ and hence $m_i | (y - x)$ for each i. Since m_i are relatively prime this implies $M | (y - x)$, hence $y - x = 0$. Since the sets are finite and have the same size, it follows that the mapping f is also invertible, which is exactly the statement of the theorem. □

Note that the theorem is not constructive (just like the theorem about the modular inverse).

Chinese remainder algorithm

How to find the x in the Chinese remainder theorem?
Let $M_i = M/m_i$, for example $M_1 = m_2 \cdots m_k$. Let m_i' be $(M_i)^{-1}$ modulo m_i (it exists). Let

$$x = a_1 M_1 m_1' + \cdots + a_k M_k m_k' \bmod M.$$

Let us show for example $x \equiv a_1 \pmod{m_1}$. We have $a_i M_i m_i' \equiv 0 \pmod{m_1}$ for each $i > 1$, since $m_1 | M_i$.
On the other hand, $a_1 M_1 m_1' \equiv a_1 \cdot 1 \pmod{m_1}$.

Fractions in $\mathbb{Z}_m$

Look at the equation $r \equiv yt \pmod{m}$, where m, y is given. Typically there is no unique solution for r, t; however, the quotient r/t (as a rational number) is uniquely determined if r, t are required to be small compared to m.

Theorem (Rational reconstruction)

Let $r^*, t^* > 0$ and y be integers with $2r^*t^* < m$. Let us call the pair (r, t) of integers admissible if $|r| \leqslant r^*$, $0 < t \leqslant t^*$, and $r \equiv yt \pmod{m}$. Then, there is a rational number q_y such that $r/t = q_y$ for all admissible pairs (r, t).

Proof.

Suppose that both (r_1, t_1) and (r_2, t_2) are admissible pairs: we want to prove $r_1/t_1 = r_2/t_2$. We have, modulo m:

$$r_1 \equiv t_1 y,$$
$$r_2 \equiv t_2 y.$$

Linear combination gives $r_1 t_2 - r_2 t_1 \equiv 0$, hence $m | (r_1 t_2 - r_2 t_1)$. Since $m > 2r^* t^*$ this implies $r_1 t_2 = r_2 t_1$. Dividing by $t_1 t_2$ gives the result. □

Finding an admissible pair (if it exists) under the condition

$$n \geqslant 4r^* t^*,$$

by the Euclidean algorithm: see the book.

Error correction

Let $m_1, \dots, m_k$ be mutually coprime moduli, $M = m_1 \cdots m_k$. Let $0 < Z < M$ and $0 < P$ be integers. A set $B \subset \{1, \dots, k\}$ is called P-admissible if $\prod_{i \in B} m_i \leqslant P$.

Example

If $(m_1, m_2, m_3, m_4) = (2, 3, 5, 7)$ and $P = 8$ then the admissible sets are $\{1\}, \{2\}, \{1, 2\}, \{3\}, \{4\}$.

Let y be an arbitrary integer. An integer $0 \leqslant z \leqslant Z$ is called (Z, P)-admissible for y if the set of indices

$$B = \{\, i : z \not\equiv y \pmod{m_i} \,\}$$

is P-admissible. We can say y has errors compared to z in the residues $y \bmod m_i$ for $i \in B$.

An admissible z can be recovered from y, provided Z, P are small:

Theorem

If $M > 2PZ^2$ then for every y and there is at most one z that is (Z, P)-admissible for it.

Proof.

Let $t = \prod_{i \in B} m_i$. Then it is easy to see that

$$tz \equiv ty \pmod{M}$$

holds. Let $r = tz$, $r^* = PZ$, $t^* = P$, then $|tz| \leqslant r^*$ and $t \leqslant t^*$ while $M > 2r^*t^*$. The Rational Reconstruction Theorem implies therefore that $z = r/t$ is uniquely determined by y. □

If the stronger condition $M > 4P^2Z$ is required then following the book, the value z can also be found efficiently using the Euclidean algorithm.

Euler's phi function

See the definition in the book. Computing it for p, p^{α}, pq.
The multiplicative order of a residue.

Theorem (Euler)

For $a \in \mathbb{Z}_m^*$ we have $a^{\phi(m)} \equiv 1 \pmod{m}$.

Proof.

Corollary

Fermat's little theorem.

Some properties of phi

Theorem

For positive integers m, n *with* $\gcd(m, n) = 1$ *we have* $\phi(mn) = \phi(m)\phi(n)$.

Proof.

One-to-one map between $\mathbb{Z}_{mn}^*$ and $\mathbb{Z}_m^* \times \mathbb{Z}_n^*$. □

Application: formula for $\phi(n)$.

Theorem

We have $\sum_{d|n} \phi(d) = n$.

Proof.

To each $0 \leqslant k < n$ let us assign the pair (d, k') where $d = \gcd(k, n)$ and $k' = k/d$. Then for each divisor d of n, the numbers k' occurring in some (d, k') will run through each element of $\mathbb{Z}^*_{n/d}$ once, hence $\sum_{d|n} \phi(n/d) = n$. □

Modular exponentiation

In the exponents, we compute modulo $\phi(m)$.

Examples

- For prime $p > 2$ and $\gcd(a, p) = 1$, we have $a^{\frac{p-1}{2}} \equiv \pm 1$.
- For composite m, this is no more the case. If $m = pq$ with primes $p, q > 2$ then $x^2 \equiv 1$ has 4 solutions, since $x \bmod p = \pm 1$ and $x \bmod q = \pm 1$ can be independently of each other. See $p = 3, q = 5$.

Fast modular exponentiation: the repeated squaring trick.

Primitive root (generator).

Example

If g is a primitive root modulo a prime $p > 2$ then $a^{\frac{p-1}{2}} \equiv -1$.

Theorem

Primitive root exists for m if and only if $m = 2, 4, p^{\alpha}, 2p^{\alpha}$ for odd prime p.

Proof later.
When there is a primitive root, the multiplicative structure (group) $\mathbb{Z}_m^*$ is the same as (isomorphic to) the additive group $\mathbb{Z}_{\phi(m)}^+$.

Chebyshev's theorem

Binomial coefficients. The definition of $\pi(n), \vartheta(n)$.

Proposition

$$4^n/(n+1) < \binom{2n}{n} < \binom{2n+1}{n+1} < 4^n.$$

Lemma (Upper bound on $\vartheta(n)$)

We have $\vartheta(n) \leqslant 2n$.

Proof.

We have $\vartheta(2m+1) - \vartheta(m+1) \leqslant \log\binom{2m+1}{m+1} \leqslant 2m$. From here, induction using $\vartheta(2m-1) = \vartheta(2m)$. □

Proposition

$$\nu_p(n!) = \sum_{k \geqslant 1} \lfloor n/p^k \rfloor.$$

Lemma (Lower bound in $\pi(n)$)

$$\pi(n) \geqslant (1/2)n/\log n.$$

Proof.

For $N = \binom{2m}{m}$ we have

$$\nu_p(N) = \sum_{k \geqslant 1} (\lfloor 2m/p^k \rfloor - 2\lfloor m/p^k \rfloor).$$

Recall the exercise showing $0 \leqslant \lfloor 2x \rfloor - 2\lfloor x \rfloor \leqslant 1$, hence this is sum is between 0 and $\leqslant \log_p(2m)$. So,

$$\begin{aligned} m \leqslant \log N \leqslant \sum_{p \leqslant 2m} \nu_p(N) \log p &\leqslant \sum_{p \leqslant 2m} \log_p(2m) \log p \\ &= \sum_{p \leqslant 2m} \log(2m) = \pi(2m) \log(2m), \end{aligned}$$

$$(1/2)(2m)/\log(2m) \leqslant \pi(2m).$$

For odd n, note $\pi(2m-1) = \pi(2m)$ and that $x \log x$ is monotone. □

Theorem

We have $\vartheta(n) \approx \pi(n)\log n$, *that is* $\frac{\vartheta(n)}{\pi(n)\log n} \to 1$.

Proof.

$\vartheta(n) \leqslant \pi(n)\log n$ is immediate. For the lower bound, cut the sum at $p \geqslant n^{\lambda}$ for some constant $0 < \lambda < 1$. □

From all the above, we found

Theorem (Chebyshev)

We have $\pi(n) \stackrel{*}{=} \frac{n}{\log n}$.

Abelian groups

Proposition

Identity and inverse are unique.

Examples

$\mathbb{Z}^+, \mathbb{Q}^+, \mathbb{R}^+, \mathbb{C}^+, n\mathbb{Z}^+, \mathbb{Z}_n^+, \mathbb{Z}_n^*$.
$\mathbb{Q}^* \setminus \{0\}$ and $[0, \infty) \cap \mathbb{Q}^*$ for multiplication.

Examples

Non-abelian groups:

- 2×2 integer matrices with determinant ± 1
- 2×2 integer matrices with determinant 1
- All permutations of $\{1, \ldots, n\}$.

To create new groups

Cyclic groups, examples. Generators of a cyclic group.
Direct product $G_1 \times G_2$.

Example

The set of all ± 1 strings of length n with respect to termwise multiplication: this is "essentially the same" as $\mathbb{Z}_2^n$.

When is a direct product of two cyclic groups cyclic? Examples.

Subgroups

A subset closed with respect to addition and inverse. Then it is also a group.

Examples

- mG (or G^m in multiplicative notation).
- $G\{m\} = \{ g \in G : mg = 0 \}$.

Theorem

Every subgroup of $\mathbb{Z}$ is of the form $m\mathbb{Z}$.

We proved this already since subgroups of $(\mathbb{Z}, +)$ are just the ideals of $(\mathbb{Z}, +, *)$

Theorem

If H is finite then it is a subgroup already if it is closed under addition.

Creating new subgroups

$H_1 + H_2$, $H_1 \cap H_2$.

Example

Let $G = G_1 \times G_2$, $\overline{G}_1 = G_1 \times \{0_{G_2}\}$, $\overline{G}_2 = \{0_{G_1}\} \times G_2$. Then $\overline{G}_i$ are subgroups of G, and

$$\overline{G}_1 \cap \overline{G}_2 = \{0_G\}, \qquad \overline{G}_1 + \overline{G}_2 = G.$$

So in a way, the direct product can, with the sum notation, be also called the direct sum.

Congruences

$a \equiv b \pmod{H}$ if $b - a \in H$.
We have seen for rings earlier already that if H is an ideal, this is an equvalence relation and you can add congruences. The same proof shows that if H is a subgroup you can do this.
The equivalence classes $a + H$ are called cosets.

Theorem

All cosets have the same size as H.

Proof.

If $C = a + H$ then $x \mapsto a + x$ is a bijection between H and C. □

Corollary (Lagrange theorem, for commutative groups)

If G is finite and H is its subgroup then $|H|$ divides $|G|$.

Corollary

For any element a, *its order* $\operatorname{ord}_G(a)$ *is the order of the cyclic group generated by* a, *hence it divides* $|G|$ *if* $|G|$ *is finite.*
Thus, we always have $|G| \cdot a = 0$.

The quotient group

Group operation among congruence classes, just as modulo m. This is the group G/H.

Examples

- If $G = G_1 \times G_2$ then recall $\overline{G}_1, \overline{G}_2$. Each element of $G/\overline{G}_1$ can be written as $(0, g_2) + \overline{G}_1$ for some g_2. So, elements of $\overline{G}_2$ form a set of representatives for the cosets, and these representatives form a subgroup.
- $\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}_m$. The class representatives do not form a subgroup.
- $\mathbb{Z}_4/2\mathbb{Z}_4$ consists of the classes $[0] = \{0, 2\}, [1] = \{1, 3\}$. The class representatives do not form a subgroup.

Two-dimensional picture.

Isomorphism, homomorphism

Isomorphism.

Example

$\mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$. But $2\mathbb{Z}_4 \cong \mathbb{Z}_2$, $\mathbb{Z}_4/2\mathbb{Z}_4 \cong \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_4$.

Homomorphism, image, kernel.

Examples

- The multiplication map, $\mathbb{Z} \to m\mathbb{Z}$. Its kernel is $\mathbb{Z}\{m\}$.
- For $a = (a_1, a_2) \in \mathbb{Z}^2$, let $\phi_a : G \times G \to G$ be defined as $(g_1, g_2) \mapsto a_1 g_1 + a_2 g_2$.
- This also defines a homomorphism $\psi_g : \mathbb{Z}^2 \to G$, if we fix $g = (g_1, g_2) \in G^2$ and view a_1, a_2 as variable.

Properties of a homomorphism

Proposition

Let $\rho : G \to G'$ *be a homomorphism.*

(i) $\rho(0_G) = 0_{G'}$, $\rho(-g) = -\rho(g)$, $\rho(ng) = n\rho(g)$.

(ii) *For any subgroup* H *of* G, $\rho(H)$ *is a subgroup of* G'.

(iii) $\ker(\rho)$ *is a subgroup of* G.

(iv) ρ *is injective if and only if* $\ker(\rho) = \{0_G\}$.

(v) $\rho(a) = \rho(b)$ *if and only if* $a \equiv b \pmod{\ker(\rho)}$.

(vi) *For every subgroup* H' *of* G', $\rho^{-1}(H')$ *is a subgroup of* G *containing* $\ker(\rho)$.

Composition of homomorphisms. Homomorphisms into and from $G_1 \times G_2$.

Theorem

For any subgroup H of an Abelian group G, the map $\rho : G \to G/H$, where $\rho(a) = a + H$ is a surjective homomorphism, with kernel H, called the natural map from G to G/H.
Conversely, for any homomorphism ρ, the factorgroup $G/\ker(\rho)$ is isomorphic to $\rho(G)$.

Examples

- The image of the multiplication map $\mathbb{Z}_8 \to \mathbb{Z}_8$, $a \mapsto 2a$ is the subgroup $2\mathbb{Z}_8$ of $\mathbb{Z}_8$. The kernel is $\mathbb{Z}_8\{2\}$, and we have $\mathbb{Z}_8/\mathbb{Z}_8\{2\} \cong 2\mathbb{Z}_8 \cong \mathbb{Z}_4$.
- (Chinese Remainder Theorem) For $m_1, \ldots, m_k$, the map $\mathbb{Z} \to \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_k}$ given by taking the remainders modulo m_i. Surjective iff the m_i are pairvise relatively prime.

Theorem

Let H_1, H_2 *be subgroups of* G*. The the map* $\rho : H_1 \times H_2 \to H_1 + H_2$ *with* $\rho(h_1, h_2) = h_1 + h_2$ *is a surjective group homomorphism that is an isomorphism iff* $H_1 \cap H_2 = \{0\}$*.*

Cyclic groups, classification

For a generator a of cyclic G, look at homomorphism $\rho_a : \mathbb{Z} \to G$, defined by $z \mapsto za$. Then $\ker(\rho_a)$ is either $\{0\}$ or $m\mathbb{Z}$ for some m. In the first case, $G \cong \mathbb{Z}$, else $G \cong \mathbb{Z}_m$

Examples

- An element n of $\mathbb{Z}_m$ generates a subgroup of order $m/\gcd(m, n)$.
- $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ is cyclic iff $\gcd(m_1, m_2) = 1$.

Subgroups

All subgroups of $\mathbb{Z}$ are of the form $m\mathbb{Z}$.

Theorem

On subgroups of a finite cyclic group $G = \mathbb{Z}_m$:

(i) *All subgroups are of the form $dG = G\{m/d\}$ where $d|m$, and $dG \subseteq d'G$ iff $d'|d$.*

(ii) *For any divisor d of m, the number of elements of order d is $\phi(d)$.*

(iii) *For any integer n we have $nG = dG$ and $G\{n\} = G\{d\}$ where $d = \gcd(m, n)$.*

Theorem

(i) *If G is of prime order then it is cyclic.*

(ii) *Subgroups of a cyclic group are cyclic.*

(iii) *Homomorphic images of a cyclic group are cyclic.*

The exponent of an Abelian group G: the smallest $m > 0$ with $mG = \{0\}$, or 0 if there is no such $m > 0$.

Theorem

Let m be the exponent of G.

(i) *m divides $|G|$.*

(ii) *If $m \neq 0$ is then the order of every element divides it.*

(iii) *G has an element of order m.*

Theorem

(i) *If prime p divides $|G|$ then G contains an element of order p.*

(ii) *The primes dividing the exponent are the same as the primes dividing the order.*

Rings

We have introduced rings earlier, now we will learn more about them.

Example

Complex numbers: pairs (a, b) with $a, b \in \mathbb{R}$, and the known operations.
Conjugation: a ring isomorphism. Norm: $z\bar{z} = a^2 + b^2$, and its properties.

Characteristic: the exponent of the additive group.

Units and fields

An element is a unit if it has a multiplicative inverse. The set of units of ring R is denoted by R^*. This is a group.

Examples

- For $z \in \mathbb{C}$, we have $z^{-1} = \bar{z}/N(z)$.
- Units in $\mathbb{Z}$, $\mathbb{Z}_m$.
- The Gaussian integers, and units among them.
- Units in $R_1 + R_2$.

Zero divisors and integral domains

R is an integral domain if it has no zero divisors.

Examples

- When is $\mathbb{Z}_m$ an integral domain?
- When is an element of $R_1 \times R_2$ a zero divisor?

Theorem

(i) $a|b$ *implies unique quotient.*

(ii) $a|b$ *and* $b|a$ *implies they differ by a unit.*

Theorem

(i) *The characteristic of an integral domain is a prime.*

(ii) *Any finite integral domain is a field.*

(iii) *Any finite field has prime power cardinality.*

Subrings

Examples

- Gaussian integers
- $\mathbb{Q}_m$.

Polynomial rings

The ring $R[\mathtt{X}]$.
The formal polynomial versus the polynomial function. In algebra, $\mathtt{X}$ is frequently called an indeterminate to make the distinction clear.
For each $a \in R$, the substitution $\rho_a : R[\mathtt{X}] \to R$ defined by $\rho_a(f(\mathtt{X})) = f(a)$ is a ring homomorphism.

Example

$\mathbb{Z}_2[\mathtt{X}]$ is our first example of a ring with finite characteristic that is not a field.

Degree $\deg(f)$. Leading coefficient $\mathrm{lc}(f)$. Monic polynomial: when the leading coefficient is 1. Constant term.
Degree Convention: $\deg(0) = -\infty$.
$\deg(fg) \leqslant \deg(f) + \deg(g)$, equality if the leading coefficients are not zero divisors.

Proposition

If D is an integral domain then $(D[\mathtt{X}])^* = D^*$.

Warning: different polynomials can give rise to the same polynomial function. Example: $\mathtt{X}^p - \mathtt{X}$ over $\mathbb{Z}_p$ defines the 0 function.

Theorem (Division with remainder)

Let $f, g \in R[\mathtt{X}]$ with $g \neq 0_R$ and $\mathrm{lc}(g) \in R^*$. Then there is a q with

$$f = q \cdot g + r, \qquad \deg(r) < \deg(g).$$

Notice the resemblance to and difference from number division.

The long division algorithm.

Theorem

Dividing by $\mathtt{X} - \alpha$*:*

$$f(\mathtt{X}) = q \cdot (\mathtt{X} - \alpha) + f(\alpha).$$

Roots of a polynomial.

Corollary

(i) α *is a root of* $f(\mathtt{X})$ *iff* $f(\mathtt{X})$ *is divisible by* $\mathtt{X} - \alpha$.

(ii) *In an integral domain, a polynomial of degree* n *has at most* n *roots.*

Theorem

If D is an integral domain then every finite subgroup G of D^* is cyclic.

Proof.

The exponent of m of G is equal to G, since $\mathtt{X}^m - 1$ has at most m roots. By an earlier theorem, G has an element whose order is the exponent. □

Corollary

Modulo any prime p, there is a primitive root.

Ideals and homomorphisms

We defined ideals earlier, this is partly a review.
Generated ideal (a), (a, b, c). Principal ideal.
Congruence modulo an ideal. Quotient ring.

Example

Let f be a monic polynomial, consider $E = R[\mathtt{X}]/(f \cdot R[\mathtt{X}]) = R[\mathtt{X}]/(f)$.

- $\mathbb{R}[X]/(\mathtt{X}^2 + 1) \cong \mathbb{C}$.
- $\mathbb{Z}_2[\mathtt{X}]/(\mathtt{X}^2 + \mathtt{X} + 1)$. Elements are $[0], [1], [\mathtt{X}], [\mathtt{X} + 1]$. Multiplication using the rule $\mathtt{X}^2 \equiv \mathtt{X} + 1$. Since $\mathtt{X}(\mathtt{X} + 1) \equiv 1$ every element has an inverse, and E is a field of size 4.

Prime ideal: If $ab \in I$ implies $a \in I$ or $b \in I$.
Maximal ideal.

Examples

- In the ring $\mathbb{Z}$, the ideal $m\mathbb{Z}$ is a prime ideal if and only if m is prime. In this case it is also maximal.
- In the ring $\mathbb{R}[X, Y]$, the ideal (X) is prime, but not maximal. Indeed, $(X) \subsetneq (X, Y) \neq \mathbb{R}[X, Y]$

Proposition

(i) *I is prime iff R/I is an integral domain.*
(ii) *I is maximal iff R/I is a field.*

Proposition

Let $\rho : R \to R'$ be a homomorphism.

(i) *Images of subrings are subrings. Images of ideals are ideals of $\rho(R)$.*

(ii) *The kernel is an ideal. ρ is injective (an embedding) iff it is $\{0\}$.*

(iii) *The inverse image of an ideal is an ideal containing the kernel.*

Proposition

The natural map $\rho : R \to R/I$ is a homomorphism.
Isomorphism between $R/\ker(\rho)$ and $\rho(R)$.

Examples

- $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is not only a group homomorphism but also a ring homomorphism.
- The mapping for the Chinese Remainder Theorem.

Polynomial factorization and congruences

(See 17.3-4 of Shoup)
We will consider elements of $F[\mathtt{X}]$ over a field F.
The associate relation between elements of $F[\mathtt{X}]$.

Theorem

Unique factorization in $F[\mathtt{X}]$. The monic irreducible factors are unique.

The proof parallels the proof of unique factorization for integers, using division with remainder.

- Every ideal is principal.
- If f, g are relatively prime then there are s, t with

$$f \cdot s + g \cdot t = 1. \tag{2}$$

- Polynomial p is irreducible iff $p \cdot F[\mathtt{X}]$ is a prime ideal, and iff it is a maximal ideal, so iff $F[\mathtt{X}]/(p)$ is a field.
  Warning: here we cannot use counting argument (as for integers) to show the existence of the inverse. We rely on (2) directly.
- Congruences modulo a polynomial. Inverse.
- Chinese remainder theorem. Interpolation.

The following theorem is also true, but its proof is longer (see 17.8 of Shoup).

Theorem

There is unique factorization over the following rings as well:

$$\mathbb{Z}[x_1, \ldots, x_n], F[x_1, \ldots, x_n],$$

where F is an arbitrary field.

Complex and real numbers

Theorem

Every polynomial in $\mathbb{C}[\mathtt{X}]$ has a root.

We will not prove this. It implies that all irreducible polynomials in $\mathbb{C}$ have degree 1.

Theorem

Every irreducible polynomial over $\mathbb{R}$ has degree 1 or 2.

Proof.

Let $f(\mathtt{X})$ be a monic polynomial with no real roots, and let

$$f(\mathtt{X}) = (\mathtt{X} - \alpha_1) \cdots (\mathtt{X} - \alpha_n)$$

over the complex numbers. Then

$$f(\mathtt{X}) = \overline{f(\mathtt{X})} = (\mathtt{X} - \overline{\alpha}_1) \cdots (\mathtt{X} - \overline{\alpha}_n).$$

Since the factorization is unique, the conjugation just permuted the roots. All the roots are in pairs: $\beta_1, \overline{\beta}_1$, $\beta_2, \overline{\beta}_2$, and so on. We have

$$(\mathtt{X} - \beta)(\mathtt{X} - \overline{\beta}) = \mathtt{X}^2 - (\beta + \overline{\beta})\mathtt{X} + \beta\overline{\beta}.$$

Since these coefficients are their own conjugates, they are real. Thus f is the product of real polynomials of degree 2. □

Roots of unity

Complex multiplication: addition of angles.
Roots of unity form a cyclic group (as a finite subgroup of the multiplicative group of a field).
Primitive nth root of unity: a generator of this group. One such generator is the root with the smallest angle.

Proposition

If ε is a root of unity different from 1 then $\sum_{i=1}^{n} \varepsilon^i = 0$.

Fourier transform

Interpolation is particularly simple if the polynomial is evaluated at roots of unity.