Algorithmic Statistics

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Abstract— **While Kolmogorov complexity is the accepted absolute measure of information content of an individual finite object, a similarly absolute notion is needed for the relation between an individual data sample and an individual model summarizing the information in the data, for example, a finite set (or probability distribution) where the data sample typically came from. The statistical theory based on such relations between individual objects can be called algorithmic statistics, in contrast to classical statistical theory that deals with relations between probabilistic ensembles. We develop the algorithmic theory of statistic, sufficient statistic, and minimal sufficient statistic. This theory is based on two-part codes consisting of the code for the statistic (the model summarizing the regularity, the meaningful information, in the data) and the model-to-data code. In contrast to the situation in probabilistic statistical theory, the algorithmic relation of (minimal) sufficiency is an absolute relation between the individual model and the individual data sample. We distinguish implicit and explicit descriptions of the models. We give characterizations of algorithmic (Kolmogorov) minimal sufficient statistic for all data samples for both description modes—in the explicit mode under some constraints. We also strengthen and elaborate earlier results on the "Kolmogorov structure function" and "absolutely non-stochastic objects"—those rare objects for which the simplest models that summarize their relevant information (minimal sufficient statistics) are at least as complex as the objects themselves. We demonstrate a close relation between the probabilistic notions and the algorithmic ones: (i) in both cases there is an "information non-increase" law; (ii) it is shown that a function is a probabilistic sufficient statistic iff it is with high probability (in an appropriate sense) an algorithmic sufficient statistic.**

I. Introduction

STATISTICAL theory ideally considers the following problem: Given a data sample and a family of models (hypotheses), select the model that produced the data. But *a priori* it is possible that the data is atypical for the model that actually produced it, or that the true model is not present in the considered model class. Therefore we have to relax our requirements. If selection of a "true" model cannot be guaranteed by any method, then as next best choice "modeling the data" as well as possible irrespective of truth and falsehood of the resulting model may be more appropriate. Thus, we change "true" to "as well as possible." The latter we take to mean that the model expresses all significant regularity present in the data. The general setting is as follows: We carry out a probabilistic experiment of which the outcomes are governed by an unknown probability distribution P. Suppose we obtain as outcome the data sample x. Given x, we want to recover the distribution P. For certain reasons we can choose a distribution from a set of acceptable distributions only (which may or may not contain P). Intuitively, our selection criteria are that (i) x should be a "typical" outcome of the distribution selected, and (ii) the selected distribution has a "simple" description. We need to make the meaning of "typical" and "simple" rigorous and balance the requirements (i) and (ii). In probabilistic statistics one analyzes the average-case performance of the selection process. There arises the problem that for individual cases the selection performance may be bad although the performance is good on average. We embark on a systematic study of model selection where the performance is related to the individual data sample and the individual model selected. It turns out to be more straightforward to investigate models that are finite sets first, and then generalize the results to models that are probability distributions. To simplify matters, and because all discrete data can be binary coded, we consider only data samples that are finite binary strings.

This paper is one of a triad of papers dealing with the best individual model for individual data: The present paper supplies the basic theoretical underpinning by way of two-part codes, [19] derives ideal versions of applied methods (MDL) inspired by the theory, and [8] treats experimental applications thereof.

Probabilistic Statistics: In ordinary statistical theory one proceeds as follows, see for example [4]: Suppose two discrete random variables X, Y have a joint probability mass function $p(x,y)$ and marginal probability mass functions $p_1(x) = \sum_y p(x,y)$ and $p_2(y) = \sum_x p(x,y)$. Then the (probabilistic) *mutual information* $I(X;Y)$ between the joint distribution and the product distribution $p_1(x)p_2(y)$ is defined by:

$$I(X;Y) = \sum_x \sum_y p(x,y) \log \frac{p(x,y)}{p_1(x)p_2(y)}, \qquad \text{(I.1)}$$

where "log" denotes the binary logarithm. Consider a probabilistic ensemble of models, say a family of probability mass functions $\{f_\theta\}$ indexed by θ, together with a distribution p_1 over θ. This way we have a random variable Θ with outcomes in $\{f_\theta\}$ and a random variable D with outcomes in the union of domains of f_θ, and $p(\theta, d) = p_1(\theta) f_\theta(d)$. Every function $T(D)$ of a data sam-

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ple D—like the sample mean or the sample variance—is called a *statistic* of D. A statistic $T(D)$ is called *sufficient* if the probabilistic mutual information

$$I(\Theta; D) = I(\Theta; T(D)) \tag{I.2}$$

for all distributions of θ. Hence, the mutual information between parameter and data sample random variables is invariant under taking sufficient statistic and vice versa. That is to say, a statistic $T(D)$ is called sufficient for Θ if it contains all the information in D about Θ. For example, consider n tosses of a coin with unknown bias θ with outcome $D = d_1 d_2 \ldots d_n$ where $d_i \in \{0, 1\}$ $(1 \leq i \leq n)$. Given n, the number of outcomes "1" is a sufficient statistic for Θ: the statistic $T(D) = s = \sum_{i=1}^{n} d_i$. Given T, all sequences with s "1"s are equally likely independent of parameter θ: Given s, if d is an outcome of n coin tosses and $T(D) = s$ then $\Pr(d \mid T(D) = s) = \binom{n}{s}^{-1}$ and $\Pr(d \mid T(D) \neq s) = 0$. This can be shown to imply (I.2) and therefore T is a sufficient statistic for Θ. According to Fisher [5]: "The statistic chosen should summarise the whole of the relevant information supplied by the sample. This may be called the Criterion of Sufficiency ... In the case of the normal curve of distribution it is evident that the second moment is a sufficient statistic for estimating the standard deviation." Note that one cannot improve on sufficiency: for every (possibly randomized) function T we have

$$I(\Theta; D) \geq I(\Theta; T(D)), \tag{I.3}$$

that is, mutual information cannot be increased by processing the data sample in any way.

A sufficient statistic may contain information that is not relevant: for a normal distribution the sample mean is a sufficient statistic, but the pair of functions which give the mean of the even-numbered samples and the odd-numbered samples respectively, is also a sufficient statistic. A statistic $T(D)$ is a *minimal* sufficient statistic with respect to an indexed model family $\{f_\theta\}$, if it is a function of all other sufficient statistics: it contains no irrelevant information and maximally compresses the information about the model ensemble. As it happens, for the family of normal distributions the sample mean is a minimal sufficient statistic, but the sufficient statistic consisting of the mean of the even samples in combination with the mean of the odd samples is not minimal. All these notions and laws are probabilistic: they hold in an average sense.

Kolmogorov Complexity: We write *string* to mean a finite binary sequence. Other finite objects can be encoded into strings in natural ways. The Kolmogorov complexity, or algorithmic entropy, $K(x)$ of a string x is the length of a shortest binary program to compute x on a universal computer (such as a universal Turing machine). Intuitively, $K(x)$ represents the minimal amount of information required to generate x by any effective process, [10]. The conditional Kolmogorov complexity $K(x \mid y)$ of x relative to y is defined similarly as the length of a shortest program to compute x if y is furnished as an auxiliary input to the computation. This conditional definition requires a warning since different authors use the same notation but mean different things. In [2] the author writes "$K(x \mid y)$" to actually mean "$K(x \mid y, K(y))$," notationally hiding the intended supplementary auxiliary information "$K(y)$." This abuse of notation has the additional handicap that no obvious notation is left to express "$K(x \mid y)$" meaning that just "y" is given in the conditional. As it happens, "$y, K(y)$" represents more information than just "y". For example, $K(K(y) \mid y)$ can be almost as large as $\log K(y)$ by a result in [6]: For $l(y) = n$ it has an upper bound of $\log n$ for all y, and for some y's it has a lower bound of $\log n - \log \log n$. In fact, this result quantifies the undecidability of the halting problem for Turing machines—for example, if $K(K(y) \mid y) = O(1)$ for all y, then the halting problem can be shown to be decidable. This is known to be false. It is customary, [13], [6], [9], to write explicitly "$K(x \mid y)$" and "$K(x \mid y, K(y))$". Even though the difference between these two quantities is not very large, these small differences do matter in the sequel. In fact, not only the precise information itself in the conditional, but also the way it is represented, is crucial, see Subsection III-A.

The functions $K(\cdot)$ and $K(\cdot \mid \cdot)$, though defined in terms of a particular machine model, are machine-independent up to an additive constant and acquire an asymptotically universal and absolute character through Church's thesis, from the ability of universal machines to simulate one another and execute any effective process. The Kolmogorov complexity of a string can be viewed as an absolute and objective quantification of the amount of information in it. This leads to a theory of *absolute* information *contents* of *individual* objects in contrast to classical information theory which deals with *average* information *to communicate* objects produced by a *random source*. Since the former theory is much more precise, it is surprising that analogs of theorems in classical information theory hold for Kolmogorov complexity, be it in somewhat weaker form. Here our aim is to provide a similarly absolute notion for individual "sufficient statistic" and related notions borrowed from probabilistic statistics.

Two-part codes: The prefix-code of the shortest effective descriptions gives an expected code word length close to the entropy and also compresses the regular objects until all regularity is squeezed out. All shortest effective descriptions are completely random themselves, without any regularity whatsoever. The idea of a two-part code for a body of data d is natural from the perspective of Kolmogorov complexity. If d does not contain any regularity at all, then it consists of purely random data and the model is precisely that. Assume that the body of data d contains regularity. With help of a description of the regularity (a model) we can describe the data compactly. Assuming that the regularity can be represented in an effective manner (that is, by a Turing machine), we encode the data as a program for that machine. Squeezing all effective regularity out of the data, we end up with a Turing machine representing the meaningful regular information in the

data together with a program for that Turing machine representing the remaining meaningless randomness of the data. However, in general there are many ways to make the division into meaningful information and remaining random information. In a painting the represented image, the brush strokes, or even finer detail can be the relevant information, depending on what we are interested in. What we require is a rigorous mathematical condition to force a sensible division of the information at hand into a meaningful part and a meaningless part.

Algorithmic Statistics: The two-part code approach leads to a more general algorithmic approach to statistics. The algorithmic statistician's task is to select a model (described possibly by a probability distribution) for which the data is typical. In a two-part description, we describe such a model and then identify the data within the set of the typical outcomes. The best models make the two-part description as concise as the best one-part description of the data. A description of such a model is an algorithmic sufficient statistic since it summarizes all relevant properties of the data. Among the algorithmic sufficient statistics, the simplest one (an algorithmic minimal sufficient statistic) is best in accordance with Ockham's Razor since it summarizes the relevant properties of the data as concisely as possible. In probabilistic data or data subject to noise this involves separating regularity (structure) in the data from random effects.

In a restricted setting where the models are finite sets a way to proceed was suggested by Kolmogorov, attribution in [16], [3], [4]. Given data d, the goal is to identify the "most likely" finite set S of which d is a "typical" element. Finding a set of which the data is typical is reminiscent of selecting the appropriate magnification of a microscope to bring the studied specimen optimally in focus. For this purpose we consider sets S such that $d \in S$ and we represent S by the *shortest* program S^* that computes the characteristic function of S. The shortest program S^* that computes a finite set S containing d, such that the two-part description consisting of S^* and $\log |S|$ is as as short as the shortest *single* program that computes d without input, is called an *algorithmic sufficient statistic*[1] This definition is non-vacuous since there does exist a two-part code (based on the model $S_d = \{d\}$) that is as concise as the shortest single code. The description of d given S^* cannot be significantly shorter than $\log |S|$. By the theory of Martin-Löf randomness [15] this means that d is a "typical" element of S. In general there can be many algorithmic sufficient statistics for data d; a shortest among them is called an *algorithmic minimal sufficient statistic*. Note that there can be possibly more than one algorithmic minimal sufficient statistic; they are defined by, but not generally computable from, the data.

In probabilistic statistics the notion of sufficient statistic (I.2) is an average notion invariant under all probability distributions over the family of indexed models. If a statistic is not thus invariant, it is not sufficient. In contrast, in the algorithmic case we investigate the relation between the data and an individual model and therefore a probability distribution over the models is irrelevant. It is technically convenient to initially consider the simple model class of finite sets to obtain our results. It then turns out that it is relatively easy to generalize everything to the model class of computable probability distributions. That class is very large indeed: perhaps it contains every distribution that has ever been considered in statistics and probability theory, as long as the parameters are computable numbers—for example rational numbers. Thus the results are of great generality; indeed, they are so general that further development of the theory must be aimed at restrictions on this model class, see the discussion about applicability in Section VII. The theory concerning the statistics of individual data samples and models one may call *algorithmic statistics*.

Background and Related Work: At a Tallinn conference in 1973, A.N. Kolmogorov formulated the approach to an individual data to model relation, based on a two-part code separating the *structure* of a string from meaningless *random* features, rigorously in terms of Kolmogorov complexity (attribution by [16], [3]). Cover [3], [4] interpreted this approach as a (sufficient) statistic. The "statistic" of the data is expressed as a finite set of which the data is a "typical" member. Following Shen [16] (see also [20], [17], [19]), this can be generalized to computable probability mass functions for which the data is "typical." Related aspects of "randomness deficiency" (formally defined later in (IV.1)) were formulated in [11], [12] and studied in [16], [20]. Algorithmic mutual information, and the associated non-increase law, were studied in [13], [14]. Despite its evident epistemological prominence in the theory of hypothesis selection and prediction, only selected aspects of the algorithmic sufficient statistic have been studied before, for example as related to the "Kolmogorov structure function" [16], [3], and "absolutely non-stochastic objects" [16], [20], [17], [21], notions also defined or suggested by Kolmogorov at the mentioned meeting. This work primarily studies quantification of the "non-sufficiency" of an algorithmic statistic, when the latter is restricted in complexity, rather than necessary and sufficient conditions for the existence of an algorithmic sufficient statistic itself. These references obtain results for plain Kolmogorov complexity (sometimes length-conditional) up to a logarithmic error term. Especially for regular data that have low Kolmogorov complexity with respect to their length, this logarithmic error term may dominate the remaining terms and eliminate all significance. Since it is precisely the regular data that one wants to assess the meaning of, a more precise analysis as we provide is required. Here we use prefix complexity to unravel the nature of a sufficient statistic. The excellent papers of Shen [16], [17] contain the major previous results related to this work (although [17] is independent).

For the relation with inductive reasoning according to minimum description length principle see [19]. The entire approach is based on Kolmogorov complexity (also

[1] It is also called the Kolmogorov sufficient statistic.

known as algorithmic information theory). Historically, the idea of assigning to each object a probability consisting of the summed negative exponentials of the lengths of all programs computing the object, was first proposed by Solomonoff [18]. Then, the shorter programs contribute more probability than the longer ones. His aim, ultimately successful in terms of theory (see [9]) and as inspiration for developing applied versions [1], was to develop a general prediction method. Kolmogorov [10] introduced the complexity proper. The prefix-version of Kolmogorov complexity used in this paper was introduced in [13] and also treated later in [2]. For a textbook on Kolmogorov complexity, its mathematical theory, and its application to induction, see [9]. We give a definition (attributed to Kolmogorov) and results from [16] that are useful later:

Definition I.1: Let α and β be natural numbers. A finite binary string x is called (α,β)-*stochastic* if there exists a finite set $S \subseteq \{0,1\}^*$ such that

$$x \in S, \quad K(S) \leq \alpha, \quad K(x) \geq \log |S| - \beta; \tag{I.4}$$

where $|S|$ denotes the cardinality of S, and $K(\cdot)$ the (prefix-) Kolmogorov complexity. As usual, "log" denotes the binary logarithm.

The first inequality with small α means that S is "simple"; the second inequality with β is small means that x is "in general position" in S. Indeed, if x had any special property p that was shared by only a small subset Q of S, then this property could be used to single out and enumerate those elements and subsequently indicate x by its index in the enumeration. Altogether, this would show $K(x) \leq K(p) + \log |Q|$, which, for simple p and small Q would be much lower than $\log |S|$. A similar notion for computable probability distributions is as follows: Let α and β be natural numbers. A finite binary string x is called (α,β)-*quasistochastic* if there exists a computable probability distribution P such that

$$P(x) > 0, \quad K(P) \leq \alpha, \quad K(x) \geq -\log P(x) - \beta. \tag{I.5}$$

Proposition I.2: There exist constants c and C, such that for every natural number n and every finite binary string x of length n:

(a) if x is (α,β)-stochastic, then x is $(\alpha + c, \beta)$-quasistochastic; and

(b) if x is (α,β)-quasistochastic and the length of x is less than n, then x is $(\alpha + c \log n, \beta + C)$-stochastic.

Proposition I.3: (a) There exists a constant C such that, for every natural number n and every α and β with $\alpha \geq \log n + C$ and $\alpha + \beta \geq n + 4 \log n + C$, all strings of length less than n are (α,β)-stochastic.

(b) There exists a constant C such that, for every natural number n and every α and β with $2\alpha + \beta < n - 6 \log n - C$, there exist strings x of length less than n that are not (α,β)-stochastic.

Note that if we take $\alpha = \beta$ then, for some boundary in between $\frac{1}{3}n$ and $\frac{1}{2}n$, the last non-(α,β)-stochastic elements disappear if the complexity constraints are sufficiently relaxed by having α, β exceed this boundary.

Outline of this Work: First, we obtain a new Kolmogorov complexity "triangle" inequality that is useful in the later parts of the paper. We define algorithmic mutual information between two individual objects (in contrast to the probabilistic notion of mutual information that deals with random variables). We show that for every computable distribution associated with the random variables, the expectation of the algorithmic mutual information equals the probabilistic mutual information up to an additive constant that depends on the complexity of the distribution. It is known that in the probabilistic setting the mutual information (an average notion) cannot be increased by algorithmic processing. We give a new proof that this also holds in the individual setting.

We define notions of "typicality" and "optimality" of sets in relation to the given data x. Denote the shortest program for a finite set S by S^* (if there is more than one shortest program S^* is the first one in the standard effective enumeration). "Typicality" is a reciprocal relation: A set S is "typical" with respect to x if x is an element of S that is "typical" in the sense of having small *randomness deficiency* $\delta^*_S(x) = \log |S| - K(x|S^*)$ (see definition (IV.1) and discussion). That is, x has about maximal Kolmogorov complexity in the set, because it can always be identified by its position in an enumeration of S in $\log |S|$ bits. Every description of a "typical" set for the data is an algorithmic statistic.

A set S is "optimal" if the best two-part description consisting of a description of S and a straightforward description of x as an element of S by an index of size $\log |S|$ is as concise as the shortest one-part description of x. This implies that optimal sets are typical sets. Descriptions of such optimal sets are algorithmic sufficient statistics, and a shortest description among them is an algorithmic minimal sufficient statistic. The mode of description plays a major role in this. We distinguish between "explicit" descriptions and "implicit" descriptions—that are introduced in this paper as a proper restriction on the recursive enumeration based description mode. We establish range constraints of cardinality and complexity imposed by implicit (and hence explicit) descriptions for typical and optimal sets, and exhibit a concrete algorithmic minimal sufficient statistic for implicit description mode. It turns out that only the complexity of the data sample x is relevant for this implicit algorithmic minimal sufficient statistic. Subsequently we exhibit explicit algorithmic sufficient statistics, and an explicit minimal algorithmic (near-)sufficient statistic. For explicit descriptions it turns out that certain other aspects of x (its enumeration rank) apart from its complexity are a major determinant for the cardinality and complexity of that statistic. It is convenient at this point to introduce some notation:

Notation I.4: From now on, we will denote by $\stackrel{+}{<}$ an inequality to within an additive constant, and by $\stackrel{+}{=}$ the situation when both $\stackrel{+}{<}$ and $\stackrel{+}{>}$ hold. We will also use $\stackrel{*}{<}$ to denote an inequality to within an multiplicative constant factor, and $\stackrel{*}{=}$ to denote the situation when both $\stackrel{*}{<}$ and $\stackrel{*}{>}$

hold.

Let is contrast our approach with the one in [16]. The comparable case there, by (I.4), is that x is (α,β)-stochastic with $\beta = 0$ and α minimal. Then, $K(x) \geq \log|S|$ for a set S of Kolmogorov complexity α. But, if S is optimal for x, then, as we formally define it later (III.4), $K(x) \stackrel{+}{=} K(S) + \log|S|$. That is (I.4) holds with $\beta \stackrel{+}{=} -K(S)$. In contrast, for $\beta = 0$ we must have $K(S) \stackrel{+}{=} 0$ for typicality. In short, optimality of S with repect to x corresponds to (I.4) by dropping the second item and replacing the third item by $K(x) \stackrel{+}{=} \log|S| + K(S)$. "Minimality" of the algorithmic sufficient statistic S^* (the shortest program for S) corresponds to choosing S with minimal $K(S)$ in this equation. This is equivalent to (I.4) with inequalities replaced by equalities and $K(S) = \alpha = -\beta$.

We consider the functions related to (α,β)-stochasticity, and improve Shen's result on maximally non-stochastic objects. In particular, we show that for every n there are objects x of length n with complexity $K(x \mid n)$ about n such that every explicit algorithmic sufficient statistic for x has complexity about n ($\{x\}$ is such a statistic). This is the best possible. In Section V, we generalize the entire treatment to probability density distributions. In Section VI we connect the algorithmic and probabilistic approaches: While previous authors have used the name "Kolmogorov sufficient statistic" because the model appears to summarize the relevant information in the data in analogy of what the classic sufficient statistic does in a probabilistic sense, a formal justification has been lacking. We give the formal relation between the algorithmic approach to sufficient statistic and the probabilistic approach: A function is a probabilistic sufficient statistic iff it is with high probability an algorithmic θ-sufficient statistic, where an algorithmic sufficient statistic is *θ-sufficient* if it satisfies also the sufficiency criterion conditionalized on θ.

II. Kolmogorov Complexity

We give some definitions to establish notation. For introduction, details, and proofs, see [9]. We write *string* to mean a finite binary string. Other finite objects can be encoded into strings in natural ways. The set of strings is denoted by $\{0,1\}^*$. The *length* of a string x is denoted by $l(x)$, distinguishing it from the *cardinality* $|S|$ of a finite set S.

Let $x,y,z \in \mathcal{N}$, where $\mathcal{N}$ denotes the natural numbers. Identify $\mathcal{N}$ and $\{0,1\}^*$ according to the correspondence

$$(0,\epsilon),(1,0),(2,1),(3,00),(4,01),\ldots.$$

Here ϵ denotes the *empty word* '' with no letters. The *length* $l(x)$ of x is the number of bits in the binary string x. For example, $l(010) = 3$ and $l(\epsilon) = 0$.

The emphasis is on binary sequences only for convenience; observations in any alphabet can be so encoded in a way that is 'theory neutral'.

A binary string x is a *proper prefix* of a binary string y if we can write $y = xz$ for $z \neq \epsilon$. A set $\{x,y,\ldots\} \subseteq \{0,1\}^*$ is *prefix-free* if for any pair of distinct elements in the set neither is a proper prefix of the other. A prefix-free set is also called a *prefix code*. Each binary string $x = x_1x_2\ldots x_n$ has a special type of prefix code, called a *self-delimiting code*,

$$\bar{x} = 1^n 0 x_1 x_2 \ldots x_n.$$

This code is self-delimiting because we can determine where the code word $\bar{x}$ ends by reading it from left to right without backing up. Using this code we define the standard self-delimiting code for x to be $x' = \overline{l(x)}x$. It is easy to check that $l(\bar{x}) = 2n+1$ and $l(x') = n + 2\log n + 1$.

Let $\langle\cdot,\cdot\rangle$ be a standard one-one mapping from $\mathcal{N}\times\mathcal{N}$ to $\mathcal{N}$, for technical reasons chosen such that $l(\langle x,y\rangle) = l(y) + l(x) + 2l(l(x)) + 1$, for example $\langle x,y\rangle = x'y = 1^{l(l(x))}0l(x)xy$. This can be iterated to $\langle\langle\cdot,\cdot\rangle,\cdot\rangle$.

The *prefix Kolmogorov complexity*, or algorithmic entropy, $K(x)$ of a string x is the length of a shortest binary program to compute x on a universal computer (such as a universal Turing machine). For technical reasons we require that the universal machine has the property that no halting program is a proper prefix of another halting program. Intuitively, $K(x)$ represents the minimal amount of information required to generate x by any effective process. We denote the *shortest program* for x by x^*; then $K(x) = l(x^*)$. (Actually, x^* is the first shortest program for x in an appropriate standard enumeration of all programs for x such as the halting order.) The conditional Kolmogorov complexity $K(x \mid y)$ of x relative to y is defined similarly as the length of a shortest program to compute x if y is furnished as an auxiliary input to the computation. We often use $K(x \mid y^*)$, or, equivalently, $K(x \mid y, K(y))$ (trivially y^* contains the same information as the $y, K(y)$). Note that "y" in the conditional is just the information about y and apart from this does not contain information about y^* or $K(y)$. For this work the difference is crucial, see the comment in Section I.

A. Additivity of Complexity

Recall that by definition $K(x,y) = K(\langle x,y\rangle)$. Trivially, the symmetry property holds: $K(x,y) \stackrel{+}{=} K(y,x)$. Later we will use many times the "Additivity of Complexity" property

$$K(x,y) \stackrel{+}{=} K(x) + K(y \mid x^*) \stackrel{+}{=} K(y) + K(x \mid y^*). \qquad \text{(II.1)}$$

This result due to [6] can be found as Theorem 3.9.1 in [9] and has a difficult proof. It is perhaps instructive to point out that the version with just x and y in the conditionals doesn't hold with $\stackrel{+}{=}$, but holds up to additive logarithmic terms that cannot be eliminated. The conditional version needs to be treated carefully. It is

$$K(x,y \mid z) \stackrel{+}{=} K(x \mid z) + K(y \mid x, K(x \mid z), z). \qquad \text{(II.2)}$$

Note that a naive version

$$K(x,y \mid z) \stackrel{+}{=} K(x \mid z) + K(y \mid x^*, z)$$

is incorrect: taking $z = x$, $y = K(x)$, the left-hand side equals $K(x^* \mid x)$, and the right-hand side equals $K(x \mid x) + K(K(x) \mid x^*, x) \stackrel{+}{=} 0$. First, we derive a (to our knowledge) new "directed triangle inequality" that is needed later.

Theorem II.1: For all x, y, z,

$$K(x \mid y^*) \stackrel{+}{<} K(x, z \mid y^*) \stackrel{+}{<} K(z \mid y^*) + K(x \mid z^*).$$

Proof: Using (II.1), an evident inequality introducing an auxiliary object z, and twice (II.1) again:

$$\begin{aligned} K(x, z \mid y^*) &\stackrel{+}{=} K(x, y, z) - K(y) \\ &\stackrel{+}{<} K(z) + K(x \mid z^*) + K(y \mid z^*) - K(y) \\ &\stackrel{+}{=} K(y, z) - K(y) + K(x \mid z^*) \\ &\stackrel{+}{=} K(x \mid z^*) + K(z \mid y^*). \end{aligned}$$

■

This theorem has bizarre consequences. These consequences are not simple unexpected artifacts of our definitions, but, to the contrary, they show the power and the genuine contribution to our understanding represented by the deep and important mathematical relation (II.1).

Denote $k = K(y)$ and substitute $k = z$ and $K(k) = x$ to find the following counterintuitive corollary: To determine the complexity of the complexity of an object y it suffices to give both y and the complexity of y. This is counterintuitive since in general we cannot compute the complexity of an object from the object itself; if we could this would also solve the so-called "halting problem", [9]. This noncomputability can be quantified in terms of $K(K(y) \mid y)$ which can rise to almost $K(K(y))$ for some y—see the related discussion on notation for conditional complexity in Section I. But in the seemingly similar, but subtly different, setting below it is possible.

Corollary II.2: As above, let k denote $K(y)$. Then, $K(K(k) \mid y, k) \stackrel{+}{=} K(K(k) \mid y^*) \stackrel{+}{<} K(K(k) \mid k^*) + K(k \mid y, k) \stackrel{+}{=} 0$. We can iterate this idea. For example, the next step is that given y and $K(y)$ we can determine $K(K(K(y)))$ in $O(1)$ bits, that is, $K(K(K(k))) \mid y, k) \stackrel{+}{=} 0$.

A direct construction works according to the following idea (where we ignore some important details): From k^* one can compute $\langle k, K(k)\rangle$ since k^* is by definition the shortest program for k and also by definition $l(k^*) = K(k)$. Conversely, from $k, K(k)$ one can compute k^*: by running of all programs of length at most $K(k)$ in dovetailed fashion until the first programme of length $K(k)$ halts with output k; this is k^*. The shortest program that computes the pair $\langle y, k\rangle$ has length $\stackrel{+}{=} k$: We have $K(y, k) \stackrel{+}{=} k$ (since the shortest program y^* for y carries both the information about y and about $k = l(y^*)$). By (II.1) therefore $K(k) + K(y \mid k, K(k)) \stackrel{+}{=} k$. In view of the information equivalence of $\langle k, K(k)\rangle$ and k^*, therefore $K(k) + K(y \mid k^*) \stackrel{+}{=} k$. Let r be a program of length $l(r) = K(y \mid k^*)$ that computes y from k^*. Then, since $l(k^*) = K(k)$, there is a shortest program $y^* = qk^*r$ for y where q is a fixed $O(1)$ bit self-delimiting program that unpacks and uses k^* and r to compute y. We are now in the position to show $K(K(k) \mid y, k) \stackrel{+}{=} 0$. There is a fixed $O(1)$-bit program, that includes knowledge of q, and that enumerates two lists in parallel, each in dovetailed fashion: Using k it enumerates a list of all programs that compute k, including k^*. Given y and k it enumerates another list of all programs of length $k \stackrel{+}{=} l(y^*)$ that compute y. One of these programs is $y^* = qk^*r$ that starts with qk^*. Since q is known, this self-delimiting program k^*, and hence its length $K(k)$, can be found by matching every element in the k-list with the prefixes of every element in the y list in enumeration order.

B. Information Non-Increase

If we want to find an appropriate model fitting the data, then we are concerned with the information in the data about such models. Intuitively one feels that the information in the data about the appropriate model cannot be increased by any algorithmic or probabilistic process. Here, we rigorously show that this is the case in the algorithmic statistics setting: the information in one object about another cannot be increased by any deterministic algorithmic method by more than a constant. With added randomization this holds with overwhelming probability. We use the triangle inequality of Theorem II.1 to recall, and to give possibly new proofs, of this information non-increase; for more elaborate but hard-to-follow versions see [13], [14].

We need the following technical concepts. Let us call a nonnegative real function $f(x)$ defined on strings a *semimeasure* if $\sum_x f(x) \leq 1$, and a *measure* (a probability distribution) if the sum is 1. A function $f(x)$ is called *lower semicomputable* if there is a rational valued computable function $g(n, x)$ such that $g(n+1, x) \geq g(n, x)$ and $\lim_{n\to\infty} g(n, x) = f(x)$. For an *upper semicomputable* function f we require that $-f$ is lower semicomputable. It is computable when it is both lower and upper semicomputable. (A lower semicomputable measure is also computable.)

To define the algorithmic mutual information between two individual objects x and y with no probabilities involved, it is instructive to first recall the probabilistic notion (I.1) Rewriting (I.1) as

$$\sum_x \sum_y p(x, y)[-\log p(x) - \log p(y) + \log p(x, y)],$$

and noting that $-\log p(s)$ is very close to the length of the prefix-free Shannon-Fano code for s, we are led to the following definition. [2] The *information in y about x* is defined as

$$I(y : x) = K(x) - K(x \mid y^*) \stackrel{+}{=} K(x) + K(y) - K(x, y), \tag{II.3}$$

[2]The Shannon-Fano code has nearly optimal expected code length equal to the entropy with respect to the distribution of the source [4]. However, the prefix-free code with code word length $K(s)$ has both about expected optimal code word length and individual optimal effective code word length, [9].

where the second equality is a consequence of (II.1) and states that this information is symmetrical, $I(x:y) \stackrel{+}{=} I(y:x)$, and therefore we can talk about *mutual information*.[3]

Remark II.3: The conditional mutual information is

$$\begin{aligned} I(x:y \mid z) &= K(x \mid z) - K(x \mid y, K(y \mid z), z) \\ &\stackrel{+}{=} K(x \mid z) + K(y \mid z) - K(x, y \mid z). \end{aligned}$$

$\diamondsuit$

It is important that the expectation of the algorithmic mutual information $I(x:y)$ is close to the probabilistic mutual information $I(X;Y)$—if this were not the case then the algorithmic notion would not be a sharpening of the probabilistic notion to individual objects, but something else.

Lemma II.4: Given a computable joint probability mass distribution $p(x,y)$ over (x,y) we have

$$\begin{aligned} I(X;Y) - K(p) &\stackrel{+}{<} \sum_x \sum_y p(x,y) I(x:y) \qquad \text{(II.4)} \\ &\stackrel{+}{<} I(X;Y) + 2K(p), \end{aligned}$$

where $K(p)$ is the length of the shortest prefix-free program that computes $p(x,y)$ from input (x,y).

Remark II.5: Above we required $p(\cdot,\cdot)$ to be computable. Actually, we only require that p be a lower semicomputable function, which is a weaker requirement than recursivity. However, together with the condition that $p(\cdot,\cdot)$ is a probability distribution, $\sum_{x,y} p(x,y) = 1$, this means that $p(\cdot,\cdot)$ is computable, [9], Section 8.1. $\diamondsuit$

Proof: Rewrite the expectation

$$\begin{aligned} \sum_x \sum_y p(x,y) I(x:y) &\stackrel{+}{=} \sum_x \sum_y p(x,y)[K(x) \\ &\quad + K(y) - K(x,y)]. \end{aligned}$$

Define $\sum_y p(x,y) = p_1(x)$ and $\sum_x p(x,y) = p_2(y)$ to obtain

$$\begin{aligned} \sum_x \sum_y p(x,y) I(x:y) &\stackrel{+}{=} \sum_x p_1(x) K(x) + \sum_y p_2(y) K(y) \\ &\quad - \sum_{x,y} p(x,y) K(x,y). \end{aligned}$$

Given the program that computes p, we can approximate $p_1(x)$ by a $q_1(x,y_0) = \sum_{y \le y_0} p(x,y)$, and similarly for p_2. That is, the distributions p_i $(i = 1,2)$ are lower semicomputable, and by Remark II.5, therefore, they are computable. It is known that for every computable probability mass function q we have $H(q) \stackrel{+}{<} \sum_x q(x)K(x) \stackrel{+}{<} H(q) + K(q)$, [9], Section 8.1.

Hence, $H(p_i) \stackrel{+}{<} \sum_x p_i(x)K(x) \stackrel{+}{<} H(p_i) + K(p_i)$ $(i = 1,2)$, and $H(p) \stackrel{+}{<} \sum_{x,y} p(x,y)K(x,y) \stackrel{+}{<} H(p) + K(p)$. On the other hand, the probabilistic mutual information (I.1) is expressed in the entropies by $I(X;Y) = H(p_1) + H(p_2) - H(p)$. By construction of the q_i's above, we have $K(p_1), K(p_2) \stackrel{+}{<} K(p)$. Since the complexities are positive, substitution establishes the lemma. ■

Can we get rid of the $K(p)$ error term? The answer is affirmative; by putting $p(\cdot)$ in the conditional we even get rid of the computability requirement.

Lemma II.6: Given a joint probability mass distribution $p(x,y)$ over (x,y) (not necessarily computable) we have

$$I(X;Y) \stackrel{+}{=} \sum_x \sum_y p(x,y) I(x:y \mid p),$$

where the auxiliary p means that we can directly access the values $p(x,y)$ on the auxiliary conditional information tape of the reference universal prefix machine.

Proof: The lemma follows from the definition of conditional algorithic mutual information, Remark II.3, if we show that $\sum_x p(x)K(x \mid p) \stackrel{+}{=} H(p)$, where the $O(1)$ term implicit in the $\stackrel{+}{=}$ sign is independent of p.

Equip the reference universal prefix machine, with an $O(1)$ length program to compute a Shannon-Fano code from the auxiliary table of probabilities. Then, given an input r, it can determine whether r is the Shannon-Fano code word for some x. Such a code word has length $\stackrel{+}{=} -\log p(x)$. If this is the case, then the machine outputs x, otherwise it halts without output. Therefore, $K(x \mid p) \stackrel{+}{<} -\log p(x)$. This shows the upper bound on the expected prefix complexity. The lower bound follows as usual from the Noiseless Coding Theorem. ■

We prove a strong version of the information nonincrease law under deterministic processing (later we need the attached corollary):

Theorem II.7: Given x and z, let q be a program computing z from x^*. Then

$$I(z:y) \stackrel{+}{<} I(x:y) + K(q). \qquad \text{(II.5)}$$

Proof: By the triangle inequality,

$$\begin{aligned} K(y \mid x^*) &\stackrel{+}{<} K(y \mid z^*) + K(z \mid x^*) \\ &\stackrel{+}{=} K(y \mid z^*) + K(q). \end{aligned}$$

Thus,

$$\begin{aligned} I(x:y) &= K(y) - K(y \mid x^*) \\ &\stackrel{+}{>} K(y) - K(y \mid z^*) - K(q) \\ &= I(z:y) - K(q). \end{aligned}$$

■

This also implies the slightly weaker but intuitively more appealing statement that the mutual information between strings x and y cannot be increased by processing x and y separately by deterministic computations.

Corollary II.8: Let f, g be recursive functions. Then

$$I(f(x):g(y)) \stackrel{+}{<} I(x:y) + K(f) + K(g). \qquad \text{(II.6)}$$

Proof: It suffices to prove the case $g(y) = y$ and apply it twice. The proof is by replacing the program q that computes a particular string z from a particular x^* in (II.5).

[3] The notation of the algorithmic (individual) notion $I(x:y)$ distinguishes it from the probabilistic (average) notion $I(X;Y)$. We deviate slightly from [9] where $I(y:x)$ is defined as $K(x) - K(x \mid y)$.

There, q possibly depends on x^* and z. Replace it by a program q_f that first computes x from x^*, followed by computing a recursive function f, that is, q_f is independent of x. Since we only require an $O(1)$-length program to compute x from x^* we can choose $l(q_f) \stackrel{+}{=} K(f)$.

By the triangle inequality,

$$\begin{aligned} K(y \mid x^*) &\stackrel{+}{<} K(y \mid f(x)^*) + K(f(x) \mid x^*) \\ &\stackrel{+}{=} K(y \mid f(x)^*) + K(f). \end{aligned}$$

Thus,

$$\begin{aligned} I(x:y) &= K(y) - K(y \mid x^*) \\ &\stackrel{+}{>} K(y) - K(y \mid f(x)^*) - K(f) \\ &= I(f(x):y) - K(f). \end{aligned}$$

■

It turns out that furthermore, randomized computation can increase information only with negligible probability. Let us define the *universal probability* $\mathbf{m}(x) = 2^{-K(x)}$. This function is known to be maximal within a multiplicative constant among lower semicomputable semimeasures. So, in particular, for each computable measure $\nu(x)$ we have $\nu(x) \stackrel{*}{<} \mathbf{m}(x)$, where the constant factor in $\stackrel{*}{<}$ depends on ν. This property also holds when we have an extra parameter, like y^*, in the condition.

Suppose that z is obtained from x by some randomized computation. The probability $p(z \mid x)$ of obtaining z from x is a semicomputable distribution over the z's. Therefore it is upperbounded by $\mathbf{m}(z \mid x) \stackrel{*}{<} \mathbf{m}(z \mid x^*) = 2^{-K(z|x^*)}$. The information increase $I(z:y) - I(x:y)$ satisfies the theorem below.

Theorem II.9: For all x, y, z we have

$$\mathbf{m}(z \mid x^*) 2^{I(z:y)-I(x:y)} \stackrel{*}{<} \mathbf{m}(z \mid x^*, y, K(y \mid x^*)).$$

Remark II.10: For example, the probability of an increase of mutual information by the amount d is $\stackrel{*}{<} 2^{-d}$. The theorem implies $\sum_z \mathbf{m}(z \mid x^*) 2^{I(z:y)-I(x:y)} \stackrel{*}{<} 1$, the $\mathbf{m}(\cdot \mid x^*)$-expectation of the exponential of the increase is bounded by a constant. ♢

Proof: We have

$$\begin{aligned} I(z:y) - I(x:y) &= K(y) - K(y \mid z^*) - (K(y) - K(y \mid x^*)) \\ &= K(y \mid x^*) - K(y \mid z^*). \end{aligned}$$

The negative logarithm of the left-hand side in the theorem is therefore

$$K(z \mid x^*) + K(y \mid z^*) - K(y \mid x^*).$$

Using Theorem II.1, and the conditional additivity (II.2), this is

$$\stackrel{+}{>} K(y, z \mid x^*) - K(y \mid x^*) \stackrel{+}{=} K(z \mid x^*, y, K(y \mid x^*)).$$

■

III. Finite Set Models

For convenience, we initially consider the *model class* consisting of the family of finite sets of finite binary strings, that is, the set of subsets of $\{0,1\}^*$.

A. Finite Set Representations

Although all finite sets are recursive there are different ways to represent or specify the set. We only consider ways that have in common a method of recursively enumerating the elements of the finite set one by one, and differ in knowledge of its size. For example, we can specify a set of natural numbers by giving an explicit table or a decision procedure for membership and a bound on the largest element, or by giving a recursive enumeration of the elements together with the number of elements, or by giving a recursive enumeration of the elements together with a bound on the running time. We call a representation of a finite set S *explicit* if the size $|S|$ of the finite set can be computed from it. A representation of S is *implicit* if the logsize $\lfloor \log |S| \rfloor$ can be computed from it.

Example III.1: In Section III-D, we will introduce the set S^k of strings whose elements have complexity $\leq k$. It will be shown that this set can be represented implicitly by a program of size $K(k)$, but can be represented explicitly only by a program of size k. ♢

Such representations are useful in two-stage encodings where one stage of the code consists of an index in S of length $\stackrel{+}{=} \log |S|$. In the implicit case we know, within an additive constant, how long an index of an element in the set is.

We can extend the notion of Kolmogorov complexity from finite binary strings to finite sets: The (prefix-) complexity $K_X(S)$ of a finite set S is defined by

$$\begin{aligned} K_X(S) = \min_i \{ K(i) : &\text{Turing machine } T_i \text{ computes } S \\ &\text{in representation format } X \}, \end{aligned}$$

where X is for example "implicit" or "explicit". In general S^* denotes the first shortest self-delimiting binary program ($l(S^*) = K(S)$) in enumeration order from which S can be computed. These definitions depend, as explained above, crucial on the representation format X: the way S is supposed to be represented as output of the computation can make a world of difference for S^* and $K(S)$. Since the representation format will be clear from the context, and to simplify notation, we drop the subscript X. To complete our discussion: the worst case of representation format X, a recursively enumerable representation where nothing is known about the size of the finite set, would lead to indices of unknown length. We do not consider this case.

We may use the notation

$$S_{\text{impl}}, S_{\text{expl}}$$

for some implicit and some explicit representation of S. When a result applies to both implicit and explicit representations, or when it is clear from the context which representation is meant, we will omit the subscript.

B. Optimal Model and Sufficient Statistic

In the following we will distinguish between "models" that are finite sets, and the "shortest programs" to compute those models that are finite strings. Such a shortest program is in the proper sense a statistic of the data sample as defined before. In a way this distinction between "model" and "statistic" is artificial, but for now we prefer clarity and unambiguousness in the discussion.

Consider a string x of length n and prefix complexity $K(x) = k$. We identify the *structure* or *regularity* in x that are to be summarized with a set S of which x is a *random* or *typical* member: given S (or rather, an (implicit or explicit) shortest program S^* for S), x cannot be described significantly shorter than by its maximal length index in S, that is, $K(x \mid S^*) \stackrel{+}{>} \log |S|$. Formally,

Definition III.2: Let $\beta \geq 0$ be an agreed upon, fixed, constant. A finite binary string x is a *typical* or *random* element of a set S of finite binary strings if $x \in S$ and

$$K(x \mid S^*) \geq \log |S| - \beta, \tag{III.1}$$

where S^* is an implicit or explicit shortest program for S. We will not indicate the dependence on β explicitly, but the constants in all our inequalities ($\stackrel{+}{<}$) will be allowed to be functions of this β.

This definition requires a finite S. In fact, since $K(x \mid S^*) \stackrel{+}{<} K(x)$, it limits the size of S to $O(2^k)$ and the shortest program S^* from which S can be computed) is an *algorithmic statistic* for x iff

$$K(x \mid S^*) \stackrel{+}{=} \log |S|. \tag{III.2}$$

Note that the notions of optimality and typicality are not absolute but depend on fixing the constant implicit in the $\stackrel{+}{=}$. Depending on whether S^* is an implicit or explicit program, our definition splits into implicit and explicit typicality.

Example III.3: Consider the set S of binary strings of length n whose every odd position is 0. Let x be an element of this set in which the subsequence of bits in even positions is an incompressible string. Then S is explicitly as well as implicitly typical for x. The set $\{x\}$ also has both these properties. ♢

Remark III.4: It is not clear whether explicit typicality implies implicit typicality. Section IV will show some examples which are implicitly very non-typical but explicitly at least nearly typical. ♢

There are two natural measures of suitability of such a statistic. We might prefer either the simplest set, or the largest set, as corresponding to the most likely structure 'explaining' x. The singleton set $\{x\}$, while certainly a statistic for x, would indeed be considered a poor explanation. Both measures relate to the optimality of a two-stage description of x using S:

$$\begin{aligned} K(x) \leq K(x, S) &\stackrel{+}{=} K(S) + K(x \mid S^*) \\ &\stackrel{+}{<} K(S) + \log |S|, \end{aligned} \tag{III.3}$$

where we rewrite $K(x, S)$ by (II.1). Here, S can be understood as either $S_{\rm impl}$ or $S_{\rm expl}$. Call a set S (containing x) for which

$$K(x) \stackrel{+}{=} K(S) + \log |S|, \tag{III.4}$$

optimal. Depending on whether $K(S)$ is understood as $K(S_{\rm impl})$ or $K(S_{\rm expl})$, our definition splits into implicit and explicit optimality. Mindful of our distinction between a finite set S and a program that describes S in a required representation format, we call a shortest program for an optimal set with respect to x an *algorithmic sufficient statistic* for x. Furthermore, among optimal sets, there is a direct trade-off between complexity and logsize, which together sum to $\stackrel{+}{=} k$. Equality (III.4) is the algorithmic equivalent dealing with the relation between the individual sufficient statistic and the individual data sample, in contrast to the probabilistic notion (I.2).

Example III.5: The following restricted model family illustrates the difference between the algorithmic individual notion of sufficient statistic and the probabilistic averaging one. Following the discussion in section VII, this example also illustrates the idea that the semantics of the model class should be obtained by a restriction on the family of allowable models, after which the (minimal) sufficient statistic identifies the most appropriate model in the allowable family and thus optimizes the parameters in the selected model class. In the algorithmic setting we use all subsets of $\{0,1\}^n$ as models and the shortest programs computing them from a given data sample as the statistic. Suppose we have background information constraining the family of models to the $n+1$ finite sets $S_s = \{x \in \{0,1\}^n : x = x_1 \ldots x_n \& \sum_{i=1}^n x_i = s\}$ $(0 \leq s \leq n)$. Assume that our model family is the family of Bernoulli distributions. Then, in the probabilistic sense for every data sample $x = x_1 \ldots x_n$ there is only one natural sufficient statistic: for $\sum_i x_i = s$ this is $T(x) = s$ with the corresponding model S_s. In the algorithmic setting the situation is more subtle. (In the following example we use the complexities conditional on n.) For $x = x_1 \ldots x_n$ with $\sum_i x_i = \frac{n}{2}$ taking $S_{\frac{n}{2}}$ as model yields $|S_{\frac{n}{2}}| = \binom{n}{\frac{n}{2}}$, and therefore $\log |S_{\frac{n}{2}}| \stackrel{+}{=} n - \frac{1}{2}\log n$. The sum of $K(S_{\frac{n}{2}}|n) \stackrel{+}{=} 0$ and the logarithmic term gives $\stackrel{+}{=} n - \frac{1}{2}\log n$ for the right-hand side of (III.4). But taking $x = 1010\ldots10$ yields $K(x \mid n) \stackrel{+}{=} 0$ for the left-hand side. Thus, there is no algorithmic sufficient statistic for the latter x in this model class, while every x of length n has a probabilistic sufficient statistic in the model class. In fact, the restricted model class has algorithmic sufficient statistic for data samples x of length n that have maximal complexity with respect to the frequency of "1"s, the other data samples have no algorithmic sufficient statistic in this model class. ♢

Example III.6: It can be shown that the set S of Example III.3 is also optimal, and so is $\{x\}$. Typical sets form a much wider class than optimal ones: $\{x, y\}$ is still typical for x but with most y, it will be too complex to be optimal for x.

For a perhaps less artificial example, consider complexities conditional on the length n of strings. Let y be a random string of length n, let S_y be the set of strings of length n which have 0's exactly where y has, and let x be a random element of S_y. Then x is a string random with respect to the distribution in which 1's are chosen independently with probability 0.25, so its complexity is much less than n. The set S_y is typical with respect to x but is too complex to be optimal, since its (explicit or implicit) complexity conditional on n is n. ♢

It follows that (programs for) optimal sets are statistics. Equality (III.4) expresses the conditions on the algorithmic individual relation between the data and the sufficient statistic. Later we demonstrate that this relation implies that the probabilistic optimality of mutual information (I.1) holds for the algorithmic version in the expected sense.

An algorithmic sufficient statistic $T(\cdot)$ is a sharper individual notion than a probabilistic sufficient statistic. An optimal set S associated with x (the shortest program computing S is the corresponding sufficient statistic associated with x) is chosen such that x is maximally random with respect to it. That is, the information in x is divided in a relevant structure expressed by the set S, and the remaining randomness with respect to that structure, expressed by x's index in S of $\log |S|$ bits. The shortest program for S is itself alone an algorithmic definition of structure, without a probabilistic interpretation.

One can also consider notions of *near*-typical and *near*-optimal that arise from replacing the β in (III.1) by some slowly growing functions, such as $O(\log l(x))$ or $O(\log k)$ as in [16], [17].

In [16], [20], a function of k and x is defined as the lack of typicality of x in sets of complexity at most k, and they then consider the minimum k for which this function becomes $\stackrel{+}{=} 0$ or very small. This is equivalent to our notion of a typical set. See the discussion of this function in Section IV. In [3], [4], only optimal sets are considered, and the one with the shortest program is identified as the *algorithmic minimal sufficient statistic* of x. Formally, this is the shortest program that computes a finite set S such that (III.4) holds.

C. Properties of Sufficient Statistic

We start with a sequence of lemmas that will be used in the later theorems. Several of these lemmas have two versions: for implicit sets and for explicit sets. In these cases, S will denote $S_{\rm impl}$ or $S_{\rm expl}$ respectively.

Below it is shown that the mutual information between every typical set and the data is not much less than $K(K(x))$, the complexity of the complexity $K(x)$ of the data x. For optimal sets it is at least that, and for algorithmic minimal statistic it is equal to that. The number of elements of a typical set is determined by the following:

Lemma III.7: Let $k = K(x)$. If a set S is (implicitly or explicitly) typical for x then $I(x : S) \stackrel{+}{=} k - \log |S|$.

Proof: By definition $I(x : S) \stackrel{+}{=} K(x) - K(x \mid S^*)$ and by typicality $K(x \mid S^*) \stackrel{+}{=} \log |S|$. ■

Typicality, optimality, and minimal optimality successively restrict the range of the cardinality (and complexity) of a corresponding model for a data x. The above lemma states that for (implicitly or explicitly) typical S the cardinality $|S| = \Theta(2^{k-I(x:S)})$. The next lemma asserts that for implicitly typical S the value $I(x : S)$ can fall below $K(k)$ by no more than an additive logarithmic term.

Lemma III.8: Let $k = K(x)$. If a set S is (implicitly or explicitly) typical for x then $I(x : S) \stackrel{+}{>} K(k) - K(I(x : S))$ and $\log |S| \stackrel{+}{<} k - K(k) + K(I(x : S))$. (Here, S is understood as $S_{\rm impl}$ or $S_{\rm expl}$ respectively.)

Proof: Writing $k = K(x)$, since

$$k \stackrel{+}{=} K(k, x) \stackrel{+}{=} K(k) + K(x \mid k^*) \tag{III.5}$$

by (II.1), we have $I(x : S) \stackrel{+}{=} K(x) - K(x \mid S^*) \stackrel{+}{=} K(k) - [K(x \mid S^*) - K(x \mid k^*)]$. Hence, it suffices to show $K(x \mid S^*) - K(x \mid k^*) \stackrel{+}{<} K(I(x : S))$. Now, from an implicit description S^* we can find the value $\stackrel{+}{=} \log |S| \stackrel{+}{=} k - I(x : S)$. To recover k we only require an extra $K(I(x : S))$ bits apart from S^*. Therefore, $K(k \mid S^*) \stackrel{+}{<} K(I(x : S))$. This reduces what we have to show to $K(x \mid S^*) \stackrel{+}{<} K(x \mid k^*) + K(k \mid S^*)$ which is asserted by Theorem II.1. ■

The term $I(x : S)$ is at least $K(k) - 2 \log K(k)$ where $k = K(x)$. For x of length n with $k \stackrel{+}{>} n$ and $K(k) \stackrel{+}{>} l(k) \stackrel{+}{>} \log n$, this yields $I(x : S) \stackrel{+}{>} \log n - 2 \log \log n$.

If we further restrict typical sets to optimal sets then the possible number of elements in S is slightly restricted. First we show that implicit optimality of a set with respect to a data is equivalent to typicality with respect to the data combined with effective constructability (determination) from the data.

Lemma III.9: A set S is (implicitly or explicitly) optimal for x iff it is typical and $K(S \mid x^*) \stackrel{+}{=} 0$.

Proof: A set S is optimal iff (III.3) holds with equalities. Rewriting $K(x, S) \stackrel{+}{=} K(x) + K(S \mid x^*)$ the first inequality becomes an equality iff $K(S \mid x^*) \stackrel{+}{=} 0$, and the second inequality becomes an equality iff $K(x \mid S^*) \stackrel{+}{=} \log |S|$ (that is, S is a typical set). ■

Lemma III.10: Let $k = K(x)$. If a set S is (implicitly or explicitly) optimal for x, then $I(x : S) \stackrel{+}{=} K(S) \stackrel{+}{>} K(k)$ and $\log |S| \stackrel{+}{<} k - K(k)$.

Proof: If S is optimal for x, then $k = K(x) \stackrel{+}{=} K(S) + K(x \mid S^*) \stackrel{+}{=} K(S) + \log |S|$. From S^* we can find both $K(S) \stackrel{+}{=} l(S^*)$ and $|S|$ and hence k, that is, $K(k) \stackrel{+}{<} K(S)$. We have $I(x : S) \stackrel{+}{=} K(S) - K(S \mid x^*) \stackrel{+}{=} K(S)$ by (II.1), Lemma III.9, respectively. This proves the first property. Substitution of $I(x : S) \stackrel{+}{>} K(k)$ in the expression of Lemma III.7 proves the second property. ■

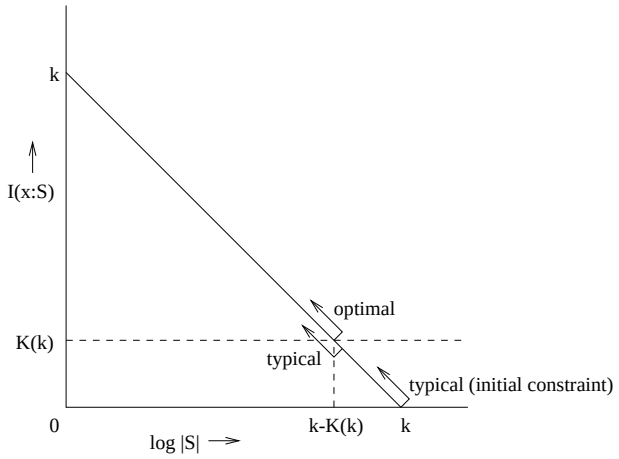

Fig. 1
RANGE OF STATISTIC ON THE STRAIGHT LINE $I(x:S) \stackrel{+}{=} K(x) - \log|S|$.

D. A Concrete Implicit Minimal Sufficient Statistic

A simplest implicitly optimal set (that is, of least complexity) is an implicit algorithmic minimal sufficient statistic. We demonstrate that $S^k = \{y : K(y) \leq k\}$, the set of all strings of complexity at most k, is such a set. First we establish the cardinality of S^k:

Lemma III.11: $\log|S^k| \stackrel{+}{=} k - K(k)$.

Proof: The lower bound is easiest. Denote by k^* of length $K(k)$ a shortest program for k. Every string s of length $k - K(k) - c$ can be described in a self-delimiting manner by prefixing it with k^*c^*, hence $K(s) \stackrel{+}{<} k - c + 2\log c$. For a large enough constant c, we have $K(s) \leq k$ and hence there are $\Omega(2^{k-K(k)})$ strings that are in S^k.

For the upper bound: by (III.5), all $x \in S^k$ satisfy $K(x \mid k^*) \stackrel{+}{<} k - K(k)$, and there can only be $O(2^{k-K(k)})$ of them. ■

From the definition of S^k it follows that it is defined by k alone, and it is the same set that is optimal for all objects of the same complexity k.

Theorem III.12: The set S^k is implicitly optimal for every x with $K(x) = k$. Also, we have $K(S^k) \stackrel{+}{=} K(k)$.

Proof: From k^* we can compute both k and $k - l(k^*) = k - K(k)$ and recursively enumerate S^k. Since also $\log|S^k| \stackrel{+}{=} k - K(k)$ (Lemma III.11), the string k^* plus a fixed program is an implicit description of S^k so that $K(k) \stackrel{+}{>} K(S^k)$. Hence, $K(x) \stackrel{+}{>} K(S^k) + \log|S^k|$ and since $K(x)$ is the shortest description by definition equality ($\stackrel{+}{=}$) holds. That is, S^k is optimal for x. By Lemma III.10 $K(S^k) \stackrel{+}{>} K(k)$ which together with the reverse inequality above yields $K(S^k) \stackrel{+}{=} K(k)$ which shows the theorem. ■

Again using Lemma III.10 shows that the optimal set S^k has least complexity among all optimal sets for x, and therefore:

Corollary III.13: The set S^k is an implicit algorithmic minimal sufficient statistic for every x with $K(x) = k$.

All algorithmic minimal sufficient statistics S for x have $K(S) \stackrel{+}{=} K(k)$, and therefore there are $O(2^{K(k)})$ of them. At least one such a statistic (S^k) is associated with every one of the $O(2^k)$ strings x of complexity k. Thus, while the idea of the algorithmic minimal sufficient statistic is intuitively appealing, its unrestricted use doesn't seem to uncover most relevant aspects of reality. The only relevant structure in the data with respect to an algorithmic minimal sufficient statistic is the Kolmogorov complexity. To give an example, an initial segment of 3.1415... of length n of complexity $\log n + O(1)$ shares the same algorithmic sufficient statistic with many (most?) binary strings of length $\log n + O(1)$.

E. Concrete Explicit (Minimal) Sufficient Statistic

Let us now consider representations of finite sets that are explicit in the sense that we can compute the cardinality of the set from the representation.

E.1 Examples of Explicit Sufficient Statistics

For example, the description program enumerates all the elements of the set and halts. Then a set like $S^k = \{y : K(y) \leq k\}$ has complexity $\stackrel{+}{=} k$ [17]: Given the program we can find an element not in S^k, which element by definition has complexity $> k$. Given S^k we can find this element and hence S^k has complexity $\stackrel{+}{>} k$. Let

$$N^k = |S^k|,$$

then by Lemma III.11 $\log N^k \stackrel{+}{=} k - K(k)$. We can list S^k given k^* and N^k which shows $K(S^k) \stackrel{+}{<} k$.

One way of implementing explicit finite representations is to provide an explicit generation time for the enumeration process. If we can generate S^k in time t recursively using k, then the previous argument shows that the complexity of every number $t' \geq t$ satisfies $K(t', k) \geq k$ so that $K(t') \stackrel{+}{>} K(t' \mid k^*) \stackrel{+}{>} k - K(k)$ by (II.1). This means that t is a huge time which as a function of k rises faster than every computable function. This argument also shows that explicit enumerative descriptions of sets S containing x by an enumerative process p plus a limit on the computation time t may take only $l(p) + K(t)$ bits (with $K(t) \leq \log t + 2\log\log t$) but $\log t$ unfortunately becomes *noncomputably large*!

In other cases the generation time is simply recursive in the input: $S_n = \{y : l(y) \leq n\}$ so that $K(S_n) \stackrel{+}{=} K(n) \leq \log n + 2\log\log n$. That is, this sufficient statistic for a random string x with $K(x) \stackrel{+}{=} n + K(n)$ has complexity $K(n)$ both for implicit descriptions and explicit descriptions: differences in complexity arise only for nonrandom strings (but not too nonrandom, for $K(x) \stackrel{+}{=} 0$ these differences vanish again).

It turns out that some strings cannot thus be explicitly represented parsimonously with low-complexity models (so that one necessarily has bad high complexity models

like S^k above). For explicit representations, Shen [16] has demonstrated the existence of a class of strings called *non-stochastic* that don't have efficient two-part representations with $K(x) \stackrel{+}{=} K(S) + \log |S|$ $(x \in S)$ with $K(S)$ significantly less than $K(x)$. See also [21]. In Section IV we improve these results to the best possible.

E.2 A Concrete Explicit Minimal Near-Sufficient Statistic

Again, consider the special set $S^k = \{y : K(y) \leq k\}$. As we have seen earlier, S^k itself cannot be explicitly optimal for x since $K(S^k) \stackrel{+}{=} k$ and $\log N^k \stackrel{+}{=} k - K(k)$, and therefore $K(S^k) + \log N^k \stackrel{+}{=} 2k - K(k)$ which considerably exceeds k. However, it turns out that a closely related set ($S^k_{m_x}$ below) is explicitly near-optimal. Let I^k_y denote the index of y in the standard enumeration of S^k, where all indexes are padded to the same length $\stackrel{+}{=} k - K(k)$ with 0's in front. For $K(x) = k$, let m_x denote the longest joint prefix of I^k_x and N^k, and let

$$I^k_x = m_x 0 i_x, \quad N^k = m_x 1 n_x.$$

Lemma III.14: For $K(x) = k$, the set $S^k_{m_x} = \{y \in S^k : m_x 0$ a prefix of $I^k_y\}$ satisfies

$$\log |S^k_{m_x}| \stackrel{+}{=} k - K(k) - l(m_x),$$
$$K(S^k_{m_x}) \stackrel{+}{<} K(k) + K(m_x) \stackrel{+}{<} K(k) + l(m_x) + K(l(m_x)).$$

Hence it is explicitly near-optimal for x (up to an addive $K(l(m_x)) \stackrel{+}{<} K(k) \stackrel{+}{<} \log k + 2 \log \log k$ term).

Proof: We can describe x by $k^* m^*_x i_x$ where $m_x 0 i_x$ is the index of x in the enumeration of S^k. Moreover, $k^* m^*_x$ explicitly describes the set $S^k_{m_x}$. Namely, using k we can recursively enumerate S^k. At some point the first string $z \in S^k_{m_x}$ is enumerated (index $I^k_z = m_x 00 \ldots 0$). By assumption $I^k_x = m_x 0 \ldots$ and $N^k = m_x 1 \ldots$. Therefore, in the enumeration of S^k eventually string u with $I^k_u = m_x 011 \ldots 1$ occurs which is the last string in the enumeration of $S^k_{m_x}$. Thus, the size of $S^k_{m_x}$ is precisely $2^{l(N^k) - l(m_x)}$, where $l(N^k) - l(m_x) \stackrel{+}{=} l(n_x) \stackrel{+}{=} \log |S^k_{m_x}|$, and $S^k_{m_x}$ is explicitly described by $k^* m^*_x$. Since $l(k^* m_x 0 i_x) \stackrel{+}{=} k$ and $\log |S^k_{m_x}| \stackrel{+}{=} k - K(k) - l(m_x)$ we have

$$K(S^k_{m_x}) + \log |S^k_{m_x}| \stackrel{+}{=} K(k) + K(m_x) + k - K(k) - l(m_x)$$
$$\stackrel{+}{=} k + K(m_x) - l(m_x) \stackrel{+}{<} k + K(l(m_x)).$$

This shows $S^k_{m_x}$ is explicitly near optimal for x (up to an additive logarithmic term). ■

Lemma III.15: Every explicit optimal set $S \subseteq S^k$ containing x satisfies

$$K(S) \stackrel{+}{>} K(k) + l(m_x) - K(l(m_x)).$$

Proof: If $S \subseteq S^k$ is explicitly optimal for x, then we can find k from S^* (as in the proof of Lemma III.10), and given k and S we find $K(k)$ as in Theorem II.1. Hence, given S^*, we can enumerate S^k and determine the maximal index I^k_y of a $y \in S$. Since also $x \in S$, the numbers I^k_y, I^k_x, N^k have a maximal common prefix m_x. Write $I^k_x = m_x 0 i_x$ with $l(i_x) \stackrel{+}{=} k - K(k) - l(m_x)$ by Lemma III.10. Given $l(m_x)$ we can determine m_x from I^k_y. Hence, from $S, l(m_x)$, and i_x we can reconstruct x. That is, $K(S) + K(l(m_x)) + l(I^k_x) - l(m_x) \stackrel{+}{>} k$, which yields the lemma. ■

Lemmas III.14, III.15 demonstrate:

Theorem III.16: The set $S^k_{m_x}$ is an explicit algorithmic minimal near-sufficient statistic for x among subsets of S^k in the following sense:

$$|K(S^k_{m_x}) - K(k) - l(m_x)| \stackrel{+}{<} K(l(m_x)),$$
$$\log |S^k_{m_x}| \stackrel{+}{=} k - K(k) - l(m_x).$$

Hence $K(S^k_{m_x}) + \log |S^k_{m_x}| \stackrel{+}{=} k \pm K(l(m_x))$. Note, $K(l(m_x)) \stackrel{+}{<} \log k + 2 \log \log k$.

E.3 Almost Always "Sufficient"

We have not completely succeeded in giving a concrete algorithmic explicit minimal sufficient statistic. However, we can show that $S^k_{m_x}$ is *almost always* minimal sufficient.

The complexity and cardinality of $S^k_{m_x}$ depend on $l(m_x)$ which will in turn depend on x. One extreme is $l(m_x) \stackrel{+}{=} 0$ which happens for the majority of x's with $K(x) = k$—for example, the first 99.9% in the enumeration order. For those x's we can replace "near-sufficient" by "sufficient" in Theorem III.16. Can the other extreme be reached? This is the case when x is enumerated close to the end of the enumeration of S^k. For example, this happens for the "non-stochastic" objects of which the existence was proven by Shen [16] (see Section IV). For such objects, $l(m_x)$ grows to $\stackrel{+}{=} k - K(k)$ and the complexity of $S^k_{m_x}$ rises to $\stackrel{+}{=} k$ while $\log |S^k_{m_x}|$ drops to $\stackrel{+}{=} 0$. That is, the explicit algorithmic minimal sufficient statistic for x is essentially x itself. For those x's we can also replace "near-sufficient" with "sufficient" in Theorem III.16. Generally: for the overwhelming majority of data x of complexity k the set $S^k_{m_x}$ is an explicit algorithmic minimal sufficient statistic among subsets of S^k (since $l(m_x) \stackrel{+}{=} 0$).

The following discussion will put what was said above into a more illuminating context. Let

$$X(r) = \{ x : l(m_x) \geq r \}.$$

The set $X(r)$ is infinite, but we can break it into slices and bound each slice separately.

Lemma III.17:

$$|X(r) \bigcap (S^k \setminus S^{k-1})| \leq 2^{-r+1} |S^k|.$$

Proof:

For every x in the set defined by the left-hand side of the inequality, we have $l(m_x) \geq r$, and the length of continuation of m_x to the total padded index of x is $\leq \lceil \log |S^k| \rceil - r \leq \log |S^k| - r + 1$. Moreover, all these

indices share the same first r bits. This proves the lemma. ■

Theorem III.18:

$$\sum_{x\in X(r)} 2^{-K(x)} \le 2^{-r+2}.$$

Proof: Let us prove first

$$\sum_{k\ge 0} 2^{-k}|S^k| \le 2. \tag{III.6}$$

By the Kraft inequality, we have, with $t_k = |S^k \setminus S^{k-1}|$,

$$\sum_{k\ge 0} 2^{-k} t_k \le 1,$$

since S^k is in 1-1 correspondence with the prefix programs of length $\le k$. Hence

$$\begin{aligned}\sum_{k\ge 0} 2^{-k}|S^k| &= \sum_{k\ge 0} 2^{-k} \sum_{i=0}^{k} t_i = \sum_{i\ge 0} t_i \sum_{k=i}^{\infty} 2^{-k} \\ &= \sum_{i\ge 0} t_i 2^{-i+1} \le 2.\end{aligned}$$

For the statement of the lemma, we have

$$\begin{aligned}\sum_{x\in X(r)} 2^{-K(x)} &= \sum_k 2^{-k} |X(r) \bigcap (S^k \setminus S^{k-1})| \\ &\le 2^{-r+1} \sum_k 2^{-k}|S^k| \le 2^{-r+2},\end{aligned}$$

where in the last inequality we used (III.6). ■

This theorem can be interpreted as follows, (we rely here on a discussion, unconnected with the present topic, about universal probability with L. A. Levin in 1973). The above theorem states $\sum_{x\in X(r)} \mathbf{m}(x) \le 2^{-r+2}$. By the multiplicative dominating property of $\mathbf{m}(x)$ with respect to every lower semicomputable semimeasure, it follows that for every computable measure ν, we have $\sum_{x\in X(r)} \nu(x) \stackrel{*}{<} 2^{-r}$. Thus, the set of objects x for which $l(m_x)$ is large has small probability with respect to every computable probability distribution.

To shed light on the exceptional nature of strings x with large $l(m_x)$ from yet another direction, let χ be the infinite binary sequence which is the characteristic function of the halting problem for our universal Turing machine: the ith bit of χ is 1 of the machine halts on the ith program, and is 0 otherwise. The expression

$$I(\chi : x) = K(x) - K(x \mid \chi)$$

shows the amount of information in the halting problem about the string x. (For an infinite sequence η, we go back formally to the definition $I(\eta : x) = K(x) - K(x \mid \eta)$ of [9], since introducing a notion of η^* in place of η here has not been shown yet to bring any benefits.) We have

$$\sum_x \mathbf{m}(x) 2^{I(\chi:x)} = \sum_x 2^{-K(x|\chi)} \le 1.$$

Therefore, if we introduce a new quantity $X'(r)$ related to $X(r)$ defined by

$$X'(r) = \{\, x : I(\chi : x) > r \,\},$$

then by Markov's inequality,

$$\sum_{x\in X'(r)} \mathbf{m}(x) 2^{I(\chi:x)} < 2^{-r}.$$

That is, the universal probability of $X'(r)$ is small. This is a new reason for $X(r)$ to be small, as is shown in the following theorem.

Theorem III.19: We have

$$I(\chi : x) \stackrel{+}{>} l(m_x) - 2\log l(m_x),$$

and (essentially equivalently) $X(r) \subset X'(r - 2\log r)$.

Remark III.20: The first item in the theorem implies: If $l(m_x) \ge r$, then $I(\chi : x) \stackrel{+}{>} r - 2\log r$. This in its turn implies the second item $X(r) \subset X'(r - 2\log r)$. Similarly, the second item essentially implies the first item. Thus, strings for which the minimal sufficient statistics has complexity much larger than $K(k)$ (that is, $l(m_x)$ is large) are exotic in the sense that they belong to the rare kind of strings about which the halting problem contains much information and *vice versa*: $I(\chi : x)$ is large. ♢

Proof: When we talk about complexity with χ in the condition, we use a Turing machine with χ as an "oracle". The theorem that $\mathbf{m}(x) = 2^{-K(x)}$ is maximal within multiplicative constant among semicomputable semimeasures is also true relative to oracles. With the help of χ, we can compute m_x, and so we can define the following new semicomputable (relative to χ) function with $c = 6/\pi^2$:

$$\nu(x \mid \chi) = c\mathbf{m}(x) 2^{l(m_x)} / l(m_x)^2.$$

We have, using III.18 and defining $Y(r) = X(r) \setminus X(r+1)$:

$$\begin{aligned}\sum_{x\in Y(r)} \nu(x \mid \chi) &= cr^{-2}2^r \sum_{x\in Y(r)} 2^{-K(x)} \\ &\le cr^{-2}2^r 2^{-r+2} \le 4cr^{-2}.\end{aligned}$$

Summing over r gives $\sum_x \nu(x \mid \chi) \le 4$, hence $\mathbf{m}(x \mid \chi) \stackrel{+}{>} \nu(x \mid \chi)$, or equivalently,

$$K(x \mid \chi) \stackrel{+}{<} -\log \nu(x \mid \chi) \stackrel{+}{=} K(x) - l(m_x) + 2\log l(m_x),$$

which proves the theorem. ■

IV. Non-Stochastic Objects

Every data sample consisting of a finite string x has an sufficient statistic in the form of the singleton set $\{x\}$. Such a sufficient statistic is not very enlightening since it simply replicates the data and has equal complexity with x. Thus, one is interested in the minimal sufficient statistic that represents the regularity, (the meaningful) information, in the data and leaves out the accidental features. This raises the

question whether every x has a minimal sufficient statistic that is significantly less complex than x itself. At a Tallinn conference in 1973 Kolmogorov (according to [16], [3]) raised the question whether there are objects x that have no minimal sufficient statistic that have relatively small complexity. In other words, he inquired into the existence of objects that are not in general position (random with respect to) any finite set of small enough complexity, that is, "absolutely non-random" objects. Clearly, such objects x have neither minimal nor maximal complexity: if they have minimal complexity then the singleton set $\{x\}$ is a minimal sufficient statistic of small complexity, and if $x \in \{0,1\}^n$ is completely incompressible (that is, it is individually random and has no meaningful information), then the uninformative universe $\{0,1\}^n$ is the minimal sufficient statistic of small complexity. To analyze the question better we need the technical notion of randomness deficiency.

Define the *randomness deficiency* of an object x with respect to a finite set S containing it as the amount by which the complexity of x as an element of S falls short of the maximal possible complexity of an element in S when S is known explicitly (say, as a list):

$$\delta_S(x) = \log |S| - K(x \mid S). \tag{IV.1}$$

The meaning of this function is clear: most elements of S have complexity near $\log |S|$, so this difference measures the amount of compressibility in x compared to the generic, typical, random elements of S. This is a generalization of the sufficiency notion in that it measures the discrepancy with typicality and hence sufficiency: if a set S is a sufficient statistic for x then $\delta_S(x) \stackrel{+}{=} 0$.

We now continue the discussion of Kolmogorov's question. Shen [16] gave a first answer by establishing the existence of absolutely non-random objects x of length n, having randomness deficiency at least $n - 2k - O(\log k)$ with respect to every finite set S of complexity $K(S) < k$ that contains x. Moreover, since the set $\{x\}$ has complexity $K(x)$ and the randomness deficiency of x with respect to this singleton set is $\stackrel{+}{=} 0$, it follows by choice of $k = K(x)$ that the complexity $K(x)$ is at least $n/2 - O(\log n)$.

Here we sharpen this result: We establish the existence of absolutely non-random objects x of length n, having randomness deficiency at least $n - k$ with respect to every finite set S of complexity $K(S \mid n) < k$ that contains x. Clearly, this is best possible since x has randomness deficiency of at least $n - K(S \mid n)$ with every finite set S containing x, in particular, with complexity $K(S \mid n)$ more than a fixed constant below n the randomness deficiency exceeds that fixed constant. That is, every sufficient statistic for x has complexity at least n. But if we choose $S = \{x\}$ then $K(S \mid n) \stackrel{+}{=} K(x \mid n) \stackrel{+}{<} n$, and, moreover, the randomness deficiency of x with respect to S is $n - K(S \mid n) \stackrel{+}{=} 0$. Together this shows that the absolutely nonrandom objects x length n of which we established the existence have complexity $K(x \mid n) \stackrel{+}{=} n$, and moreover, they have significant randomness deficiency with respect to every set S containing them that has complexity significantly below their own complexity n.

A. Kolmogorov Structure Function

We first consider the relation between the minimal unavoidable randomness deficiency of x with respect to a set S containing it, when the complexity of S is upper bounded by α. These functional relations are known as *Kolmogorov structure functions*. Kolmogorov proposed a variant of the function

$$h_x(\alpha) = \min_S \{ \log |S| : x \in S,\ K(S) < \alpha \}, \tag{IV.2}$$

where $S \subseteq \{0,1\}^*$ is a finite set containing x, the contemplated model for x, and α is a nonnegative integer value bounding the complexity of the contemplated S's. He did not specify what is meant by $K(S)$ but it was noticed immediately, as the paper [17] points out, that the behavior of $h_x(\alpha)$ is rather trivial if $K(S)$ is taken to be the complexity of a program that lists S without necessarily halting. Section III-D elaborates this point. So, the present section refers to explicit descriptions only.

It is easy to see that for every increment d we have

$$h_x(\alpha + d) \leq |h_x(\alpha) - d + O(\log d)|,$$

provided the right-hand side is non-negative, and 0 otherwise. Namely, once we have an optimal set S_α we can subdivide it in any standard way into 2^d parts and take as $S_{\alpha+d}$ the part containing x. Also, $h_x(\alpha) = 0$ implies $\alpha \stackrel{+}{>} K(x)$, and, since the choice of $S = \{x\}$ generally implies only $\alpha \stackrel{+}{<} K(x)$ is meaningful we can conclude $\alpha \stackrel{+}{=} K(x)$. Therefore it seems better advised to consider the function

$$h_x(\alpha) + \alpha - K(x) = \min_S \{ \log |S| - (K(x) - \alpha) : K(S) < \alpha \}$$

rather than (IV.2). For technical reasons related to the later analysis, we introduce the following variant of randomness deficiency (IV.1):

$$\delta^*_S(x) = \log |S| - K(x \mid S, K(S)).$$

The function $h_x(\alpha) + \alpha - K(x)$ seems related to a function of more intuitive appeal, namely $\beta_x(\alpha)$ measuring the minimal unavoidable randomness deficiency of x with respect to every finite set S, that contains it, of complexity $K(S) < \alpha$. Formally, we define

$$\beta_x(\alpha) = \min_S \{ \delta_S(x) : K(S) < \alpha \},$$

and its variant

$$\beta^*_x(\alpha) = \min_S \{ \delta^*_S(x) : K(S) < \alpha \},$$

defined in terms of δ^*_S. Note that $\beta_x(K(x)) \stackrel{+}{=} \beta^*_x(K(x)) \stackrel{+}{=} 0$. These β-functions are related to, but different from, the β in (I.4).

To compare h and β, let us confine ourselves to binary strings of length n. We will put n into the condition of all complexities.

Lemma IV.1: $\beta^*_x(\alpha \mid n) \stackrel{+}{<} h_x(\alpha \mid n) + \alpha - K(x \mid n)$.

Proof: Let $S \ni x$ be a set with $K(S \mid n) \leq \alpha$ and assume $h_x(\alpha \mid n) = \log |S|$. Tacitly understanding n in the conditions, and using the additivity property (II.1),

$$K(x) - \alpha \leq K(x) - K(S) \stackrel{+}{<} K(x,S) - K(S) \stackrel{+}{=} K(x \mid S, K(S)).$$

Therefore

$$h_x(\alpha) + \alpha - K(x) = \log |S| - (K(x) - \alpha) \stackrel{+}{>} \log |S| - K(x \mid S, K(S)) \geq \beta^*_x(\alpha).$$

■

It would be nice to have an inequality also in the other direction, but we do not know currently what is the best that can be said.

B. Sharp Bound on Non-Stochastic Objects

We are now able to formally express the notion of non-stochastic objects using the Kolmogorov structure functions $\beta_x(\alpha), \beta^*_x(\alpha)$. For every given $k < n$, Shen constructed in [16] a binary string x of length n with $K(x) \leq k$ and $\beta_x(k - O(1)) > n - 2k - O(\log k)$. Let x be one of the non-stochastic objects of which the existence is established. Substituting $k \stackrel{+}{=} K(x)$ we can contemplate the set $S = \{x\}$ with complexity $K(S) \stackrel{+}{=} k$ and x has randomness deficiency $\stackrel{+}{=} 0$ with respect to S. This yields $0 \stackrel{+}{=} \beta_x(K(x)) \stackrel{+}{>} n - 2K(x) - O(\log K(x))$. Since it generally holds that these non-stochastic objects have complexity $K(x) \stackrel{+}{>} n/2 - O(\log n)$, they are *not random, typical, or in general position* with respect to every set S containing them with complexity $K(S) \stackrel{+}{\not>} n/2 - O(\log n)$, but they are random, typical, or in general position only for sets S with complexity $K(S)$ sufficiently exceeding $n/2 - O(\log n)$ like $S = \{x\}$.

Here, we improve on this result, replacing $n - 2k - O(\log k)$ with $n - k$ and using β^* to avoid logarithmic terms. This is the best possible, since by choosing $S = \{0,1\}^n$ we find $\log |S| - K(x \mid S, K(S)) \stackrel{+}{=} n - k$, and hence $\beta^*_x(c) \stackrel{+}{<} n - k$ for some constant c, which implies $\beta^*_x(\alpha) \leq \beta_x(c) \stackrel{+}{<} n - k$ for every $\alpha > c$.

Theorem IV.2: There are constants c_1, c_2 such that for any given $k < n$ there is a a binary string x of length n with $K(x \mid n) \leq k$ such that for all $\alpha < k - c_1$ we have

$$\beta^*_x(\alpha \mid n) > n - k - c_2.$$

In the terminology of (I.4), the theorem states that there are constants c_1, c_2 such that for every $k < n$ there exists a string x of length n of complexity $K(x \mid n) \leq k$ that is not $(k - c_1, n - k - c_2)$-stochastic.

Proof: Denote the conditional universal probability as $\mathbf{m}(S \mid n) = 2^{-K(S|n)}$. We write "$S \ni x$" to indicate sets S that satisfy $x \in S$. For every n, let us define a function over all strings x of length n as follows:

$$\nu^{\leq i}(x \mid n) = \sum_{S \ni x,\ K(S|n) \leq i} \frac{\mathbf{m}(S \mid n)}{|S|} \tag{IV.3}$$

The following lemma shows that this function of x is a semimeasure.

Lemma IV.3: We have

$$\sum_x \nu^{\leq i}(x \mid n) \leq 1. \tag{IV.4}$$

Proof: We have

$$\sum_x \nu^{\leq i}(x \mid n) \leq \sum_x \sum_{S \ni x} \frac{\mathbf{m}(S \mid n)}{|S|} = \sum_S \sum_{x \in S} \frac{\mathbf{m}(S \mid n)}{|S|} = \sum_S \mathbf{m}(S \mid n) \leq 1.$$

■

Lemma IV.4: There are constants c_1, c_2 such that for some x of length n,

$$\nu^{\leq k - c_1}(x \mid n) \leq 2^{-n}, \tag{IV.5}$$

$$k - c_2 \leq K(x \mid n) \leq k. \tag{IV.6}$$

Proof: Let us fix $0 < c_1 < k$ somehow, to be chosen appropriately later. Inequality (IV.4) implies that there is an x with (IV.5). Let x be the first string of length n with this property. To prove the right inequality of (IV.6), let p be the program of length $\leq i = k - c_1$ that terminates last in the standard running of all these programs simultaneously in dovetailed fashion, on input n. We can use p and its length $l(p)$ to compute all programs of size $\leq l(p)$ that specify finite sets using n. Hence we have a list of all sets S with $K(S \mid n) \leq i$. Using it, for each y of length n we can compute $\nu^{\leq i}(y \mid n)$, by using the definition (IV.3) explicitly. Since x is defined as the first y with $\nu^{\leq i}(y \mid n) \leq 2^{-n}$, we can thus find it using p and some program of constant length. If c_1 is chosen large enough then this implies $K(x \mid n) \leq k$.

On the other hand, we have

$$\nu^{\leq K(\{x\}|n)}(x \mid n) \geq 2^{-K(\{x\}|n)}.$$

This implies, by the definition of x, that either $K(\{x\} \mid n) > k - c_1$ or $K(\{x\} \mid n) \geq n$. Since $K(x \mid n) \stackrel{+}{=} K(\{x\} \mid n))$ we get the left inequality of (IV.6) in both cases for an appropriate c_2. ■

Consider now a new semicomputable function

$$\mu_{x,i}(S \mid n) = \frac{2^n \mathbf{m}(S \mid n)}{|S|}$$

on all finite sets $S \ni x$ with $K(S \mid n) \leq i$. Then we have, with $i = k - c_1$:

$$\sum_S \mu_{x,i}(S \mid n) = 2^n \sum_{S \ni x,\ K(S|n) \leq i} \frac{\mathbf{m}(S \mid n)}{|S|} = 2^n \nu^{\leq i}(x \mid n) \leq 1$$

by (IV.5), and so, using (IV.6),

$$\begin{aligned} &K(S \mid x, K(x \mid n)) \\ &\quad \stackrel{+}{=} K(S \mid x, k) \stackrel{+}{<} -\log \mu_{x,i}(S \mid n) \\ &\quad = \log |S| - n + K(S \mid n). \end{aligned} \tag{IV.7}$$

We have, by the additivity property (II.1) and (IV.7):

$$\begin{aligned} &K(x \mid S, K(S \mid n), n) \\ &\quad \stackrel{+}{=} K(x \mid n) + K(S \mid x, K(x \mid n)) - K(S \mid n) \\ &\quad \stackrel{+}{<} k + \log |S| - n. \end{aligned}$$

Hence $\delta^*(x \mid S, n) = \log |S| - K(x \mid S, K(S \mid n), n) \stackrel{+}{>} n - k$. ∎

Let x be one of the non-stochastic objects of which the existence is established by Theorem IV.2. Choose x with $K(x \mid n) \stackrel{+}{=} k$ so that the set $S = \{x\}$ has complexity $K(S \mid n) = k - c_1$ and x has randomness deficiency $\stackrel{+}{=} 0$ with respect to S. Because x is non-stochastic, this yields $0 \stackrel{+}{=} \beta^*_x(k - c_1 \mid n) \stackrel{+}{>} n - K(x \mid n)$. For every x we have $K(x \mid n) \stackrel{+}{<} n$. Together it follows that $K(x \mid n) \stackrel{+}{=} n$. That is, these non-stochastic objects x have complexity $K(x \mid n) \stackrel{+}{=} n$. Nonetheless, there is a constant c' such that x is *not random, typical, or in general position* with respect to any explicitly represented finite set S containing it that has complexity $K(S \mid n) < n - c'$, but they are random, typical, or in general position for some sets S with complexity $K(S \mid n) \stackrel{+}{>} n$ like $S = \{x\}$. That is, every explicit sufficient statistic S for x has complexity $K(S \mid n) \stackrel{+}{=} n$, and $\{x\}$ is such a statistic.

V. Probabilistic Models

It remains to generalize the model class from finite sets to the more natural and significant setting of probability distributions. Instead of finite sets the models are computable probability density functions $P : \{0,1\}^* \to [0,1]$ with $\sum P(x) \leq 1$—we allow defective probability distributions where we may concentrate the surplus probability on a distinguished "undefined" element. "Computable" means that there is a Turing machine T_P that computes approximations to the value of P for every argument (more precise definition follows below). The (prefix-) complexity $K(P)$ of a computable partial function P is defined by

$$K(P) = \min_i \{K(i) : \text{Turing machine } T_i \text{ computes } P\}.$$

Equality (III.2) now becomes

$$K(x \mid P^*) \stackrel{+}{=} -\log P(x),$$

and equality (III.4) becomes

$$K(x) \stackrel{+}{=} K(P) - \log P(x).$$

As in the finite set case, the complexities involved are crucially dependent on what we mean by "computation" of $P(x)$, that is, on the requirements on the format in which the output is to be represented. Recall from [9] that Turing machines can compute rational numbers: If a Turing machine T computes $T(x)$, then we interpret the output as a pair of natural numbers, $T(x) = \langle p, q \rangle$, according to a standard pairing function. Then, the rational value computed by T is by definition p/q. The distinction between explicit and implicit description of P corresponding to the finite set model case is now defined as follows:

- It is *implicit* if there are positive constants c, C such that the Turing machine T computing P halts with rational value $T(x)$ with $cP(x) < T(x) < CP(x)$. Hence $-\log T(x) \stackrel{+}{=} -\log P(x)$.
- It is *explicit* if the Turing machine T computing P, given x and a tolerance ϵ halts with rational value $P(x) - \epsilon < T(x) < P(x) + \epsilon$.

The implicit and explicit descriptions of finite sets and of uniform distributions with $P(x) = 1/|S|$ for all $x \in S$ and $P(x) = 0$ otherwise, are as follows: An implicit (explicit) description of P is identical with an implicit (explicit) description of S, up to a short fixed program which indicates which of the two is intended, so that $K(P(x)) \stackrel{+}{=} K(S)$ for $P(x) > 0$ (equivalently, $x \in S$).

To complete our discussion: the worst case of representation format, a recursively enumerable approximation of $P(x)$ where nothing is known about its value, would lead to indices $-\log P(x)$ of unknown length. We do not consider this case.

The properties for the probabilistic models are loosely related to the properties of finite set models by Proposition I.2. We sharpen the relations by appropriately modifying the treatment of the finite set case, but essentially following the same course.

We may use the notation

$$P_{\text{impl}}, P_{\text{expl}}$$

for some implicit and some explicit representation of P. When a result applies to both implicit and explicit representations, or when it is clear from the context which representation is meant, we will omit the subscript.

A. Optimal Model and Sufficient Statistic

As before, we distinguish between "models" that are computable probability distributions, and the "shortest programs" to compute those models that are finite strings.

Consider a string x of length n and prefix complexity $K(x) = k$. We identify the *structure* or *regularity* in x that are to be summarized with a computable probability density function P with respect to which x is a *random* or *typical* member. For x typical for P holds the following [9]: Given an (implicitly or explicitly described) shortest program P^* for P, a shortest binary program computing x (that is, of length $K(x \mid P^*)$) can not be significantly shorter than its Shannon-Fano code [4] of length $-\log P(x)$, that is, $K(x \mid P^*) \stackrel{+}{>} -\log P(x)$. By definition, we fix some agreed upon constant $\beta \geq 0$, and require

$$K(x \mid P^*) \geq -\log P(x) - \beta.$$

As before, we will not indicate the dependence on β explicitly, but the constants in all our inequalities ($\stackrel{+}{<}$) will be allowed to be functions of this β. This definition requires a positive $P(x)$. In fact, since $K(x \mid P^*) \stackrel{+}{<} K(x)$, it limits the size of $P(x)$ to $\Omega(2^{-k})$. The shortest program P^* from which a probability density function P can be computed is an *algorithmic statistic* for x iff

$$K(x \mid P^*) \stackrel{+}{=} -\log P(x). \tag{V.1}$$

There are two natural measures of suitability of such a statistic. We might prefer either the simplest distribution, or the largest distribution, as corresponding to the most likely structure 'explaining' x. The singleton probability distribution $P(x) = 1$, while certainly a statistic for x, would indeed be considered a poor explanation. Both measures relate to the optimality of a two-stage description of x using P:

$$\begin{aligned} K(x) \leq K(x,P) &\stackrel{+}{=} K(P) + K(x \mid P^*) \\ &\stackrel{+}{<} K(P) - \log P(x), \end{aligned} \tag{V.2}$$

where we rewrite $K(x,P)$ by (II.1). Here, P can be understood as either $P_{\rm impl}$ or $P_{\rm expl}$. Call a distribution P (with positive probability $P(x)$) for which

$$K(x) \stackrel{+}{=} K(P) - \log P(x), \tag{V.3}$$

optimal. (More precisely, we should require $K(x) \geq K(P) - \log P(x) - \beta$.) Depending on whether $K(P)$ is understood as $K(P_{\rm impl})$ or $K(P_{\rm expl})$, our definition splits into implicit and explicit optimality. The shortest program for an optimal computable probability distribution is a *algorithmic sufficient statistic* for x.

B. Properties of Sufficient Statistic

As in the case of finite set models , we start with a sequence of lemmas that are used to obtain the main results on minimal sufficient statistic. Several of these lemmas have two versions: for implicit distributions and for explicit distributions. In these cases, P will denote $P_{\rm impl}$ or $P_{\rm expl}$ respectively.

Below it is shown that the mutual information between every typical distribution and the data is not much less than $K(K(x))$, the complexity of the complexity $K(x)$ of the data x. For optimal distributions it is at least that, and for algorithmic minimal statistic it is equal to that. The log-probability of a typical distribution is determined by the following:

Lemma V.1: Let $k = K(x)$. If a distribution P is (implicitly or explicitly) typical for x then $I(x : P) \stackrel{+}{=} k + \log P(x)$.

Proof: By definition $I(x : P) \stackrel{+}{=} K(x) - K(x \mid P^*)$ and by typicality $K(x \mid P^*) \stackrel{+}{=} -\log P(x)$. ■

The above lemma states that for (implicitly or explicitly) typical P the probability $P(x) = \Theta(2^{-(k - I(x:P))})$. The next lemma asserts that for implicitly typical P the value $I(x : P)$ can fall below $K(k)$ by no more than an additive logarithmic term plus the amount of information required to compute $-\log P(x)$ from P.

Lemma V.2: Let $k = K(x)$. If a distribution P is (implicitly or explicitly) typical for x then $I(x : P) \stackrel{+}{>} K(k) - K(I(x : P)) - K(-\log P(x) \mid P^*)$ and $-\log P(x) \stackrel{+}{<} k - K(k) + K(I(x : P)) + K(-\log P(x) \mid P^*)$. (Here, P is understood as $P_{\rm impl}$ or $P_{\rm expl}$ respectively.)

Proof: Writing $k = K(x)$, since

$$k \stackrel{+}{=} K(k,x) \stackrel{+}{=} K(k) + K(x \mid k^*) \tag{V.4}$$

by (II.1), we have $I(x : P) \stackrel{+}{=} K(x) - K(x \mid P^*) \stackrel{+}{=} K(k) - [K(x \mid P^*) - K(x \mid k^*)]$. Hence, it suffices to show $K(x \mid P^*) - K(x \mid k^*) \stackrel{+}{<} K(I(x : P)) + K(-\log P(x) \mid P^*)$. Now, from an implicit description P^* and a program q of length $\stackrel{+}{=} K(-\log P(x) \mid P^*)$ we can find the value $\stackrel{+}{=} -\log P(x) \stackrel{+}{=} k - I(x : P)$. To recover k, we only require an extra $K(I(x : P))$ bits apart from P^* and q. Therefore, $K(k \mid P^*) \stackrel{+}{<} K(I(x : P)) + K(-\log P(x) \mid P^*)$. This reduces what we have to show to $K(x \mid P^*) \stackrel{+}{<} K(x \mid k^*) + K(k \mid P^*)$ which is asserted by Theorem II.1. ■

Note that for distributions that are uniform (or almost uniform) on a finite support we have $K(-\log P(x) \mid P^*) \stackrel{+}{=} 0$: In this borderline case the result specializes to that of Lemma III.8 for finite set models.

On the other end of the spectrum, the given lower bound on $I(x : P)$ drops in case knowledge of P^* doesn't suffice to compute $-\log P(x)$, that is, if $K(-\log P(x) \mid P^*) \gg 0$ for an statistic P^* for x. The question is, whether we can exhibit such a probability distribution that is also computable? The answer turns out to be affirmative. By a result due to R. Solovay and P. Gács, [9] Exercise 3.7.1 on p. 225-226, there is a computable function $f(x) \stackrel{+}{>} K(x)$ such that $f(x) \stackrel{+}{=} K(x)$ for infinitely many x. Considering the case of P optimal for x (a stronger assumption than that P is just typical) we have $-\log P(x) \stackrel{+}{=} K(x) - K(P)$. Choosing $P(x)$ such that $-\log P(x) \stackrel{+}{=} \log f(x) - K(P)$, we have that $P(x)$ is computable since $f(x)$ is computable and $K(P)$ is a fixed constant. Moreover, there are infinitely many x's for which P is optimal, so $K(-\log P(x) \mid P^*) \to \infty$ for $x \to \infty$ through this special sequence.

If we further restrict typical distributions to optimal ones then the possible positive probabilities assumed by distribution P are slightly restricted. First we show that implicit optimality with respect to some data is equivalent to typicality with respect to the data combined with effective constructability (determination) from the data.

Lemma V.3: A distribution P is (implicitly or explicitly) optimal for x iff it is typical and $K(P \mid x^*) \stackrel{+}{=} 0$.

Proof: A distribution P is optimal iff (V.2) holds with equalities. Rewriting $K(x,P) \stackrel{+}{=} K(x) + K(P \mid x^*)$ the first inequality becomes an equality iff $K(P \mid x^*) \stackrel{+}{=} 0$, and the second inequality becomes an equality iff $K(x \mid P^*) \stackrel{+}{=} -\log P(x)$ (that is, P is a typical distribution). ■

Lemma V.4: Let $k = K(x)$. If a distribution P is (implicitly or explicitly) optimal for x, then $I(x : P) \stackrel{+}{=} K(P) \stackrel{+}{>} K(k) - K(-\log P(x) \mid P^*)$ and $-\log P(x) \stackrel{+}{<} k - K(k) + K(-\log P(x) \mid P^*)$.

Proof: If P is optimal for x, then $k = K(x) \stackrel{+}{=} K(P) + K(x \mid P^*) \stackrel{+}{=} K(P) - \log P(x)$. From P^* and a program q of length $K(-\log P(x) \mid P^*)$, we can find both $K(P) \stackrel{+}{=} l(P^*)$ and $-\log P(x)$, and hence k, that is, $K(k) \stackrel{+}{<} K(P) + K(-\log P(x) \mid P^*)$. We have $I(x : P) \stackrel{+}{=} K(P) - K(P \mid x^*) \stackrel{+}{=} K(P)$ by (II.1), Lemma V.3, respectively. This proves the first property. Substitution of $I(x : P) \stackrel{+}{>} K(k) - K(-\log P(x) \mid P^*)$ in the expression of Lemma V.1 proves the second property. ■

Note that for distributions that are uniform (or almost uniform) on a finite support we have $K(-\log P(x) \mid P^*) \stackrel{+}{=} 0$: In this borderline case the result specializes to that of Lemma III.10 for finite set models.

On the other end of the spectrum, we have the case that knowledge of P^* doesn't help to compute $-\log P(x)$, that is, $K(-\log P(x) \mid P^*) \gg 0$ as exemplified above. Then, the lower bound on $I(x : P) \stackrel{+}{=} K(P)$ drops towards 0 while the upper bound on $-\log P(x)$ rises towards k.

C. Concrete Minimal Sufficient Statistic

A simplest implicitly optimal distribution (that is, of least complexity) is an implicit algorithmic minimal sufficient statistic. As before, let $S^k = \{y : K(y) \leq k\}$. Define the distribution $P^k(x) = 1/|S^k|$ for $x \in S^k$, and $P^k(x) = 0$ otherwise. The demonstration that $P^k(x)$ is an implicit algorithmic minimal sufficient statistic proceeeds completely analogous to the finite set model setting, Corollary III.13, using the substitution $K(-\log P^k(x) \mid (P^k)^*) \stackrel{+}{=} 0$.

A similar equivalent construction suffices to obtain an explicit algorithmic minimal near-sufficient statistic for x, analogous to $S^k_{m_x}$ in the finite set model setting, Theorem III.16. That is, $P^k_{m_x}(y) = 1/|S^k_{m_x}|$ for $y \in S^k_{m_x}$, and 0 otherwise.

In general, one can develop the theory of minimal sufficient statistic for models that are probability distributions similarly to that of finite set models, up to the extra additive term $K(-\log P(x) \mid P^*)$. It is not known how far that term can be reduced.

D. Non-Quasistochastic Objects

As in the more restricted case of finite sets, there are objects that are not typical for any explicitly computable probability distribution that has complexity significantly below that of the object itself. With the terminology of (I.5), we may call such *absolutely non-quasistochastic*.

By Proposition I.2, item (b), there are constants c and C such that if x is not $(\alpha + c\log n, \beta + C)$-stochastic (I.4) then x is not (α, β)-quasistochastic (I.5). Substitution in Theorem IV.2 yields:

Corollary V.5: There are constants c, C such that, for every $k < n$, there are constants c_1, c_2 and a binary string x of length n with $K(x \mid n) \leq k$ such that x is not $(k - c\log n - c_1, n - k - C - c_2)$-quasistochastic.

As a particular consequence: Let x with length n be one of the non-quasistochastic strings of which the existence is established by Corollary V.5. Substituting $K(x \mid n) \stackrel{+}{<} k - c\log n$, we can contemplate the distribution $P_x(y) = 1$ for $y = x$ and and 0 otherwise. Then we have complexity $K(P_x \mid n) \stackrel{+}{=} K(x \mid n)$. Clearly, x has randomness deficiency $\stackrel{+}{=} 0$ with respect to P_x. Because of the assumption of non-quasistochasticity of x, and because the minimal randomness-deficiency $\stackrel{+}{=} n - k$ of x is always nonnegative, $0 \stackrel{+}{=} n - k \stackrel{+}{>} n - K(x \mid n) - c\log n$. Since it generally holds that $K(x \mid n) \stackrel{+}{<} n$, it follows that $n \stackrel{+}{>} K(x \mid n) \stackrel{+}{>} n - c\log n$. That is, these non-quasistochastic objects have complexity $K(x \mid n) \stackrel{+}{=} n - O(\log n)$ and are *not random, typical, or in general position* with respect to any explicitly computable distribution P with $P(x) > 0$ and complexity $K(P \mid n) \stackrel{+}{<} n - (c+1)\log n$, but they are random, typical, or in general position only for some distributions P with complexity $K(P \mid n) \stackrel{+}{>} n - c\log n$ like P_x. That is, every explicit sufficient statistic P for x has complexity $K(P \mid n) \stackrel{+}{>} n - c\log n$, and P_x is such a statistic.

VI. Algorithmic Versus Probabilistic

Algorithmic sufficient statistic, a function of the data, is so named because intuitively it expresses an individual summarizing of the relevant information in the individual data, reminiscent of the probabilistic sufficient statistic that summarizes the relevant information in a data random variable about a model random variable. Formally, however, previous authors have not established any relation. Other algorithmic notions have been successfully related to their probabilistic counterparts. The most significant one is that for every computable probability distribution, the expected prefix complexity of the objects equals the entropy of the distribution up to an additive constant term, related to the complexity of the distribution in question. We have used this property in (II.4) to establish a similar relation between the expected algorithmic mutual information and the probabilistic mutual information. We use this in turn to show that there is a close relation between the algorithmic version and the probabilistic version of sufficient statistic: A probabilistic sufficient statistic is with high probability a natural conditional form of algorithmic sufficient statistic for individual data, and, conversely, that with high probability a natural conditional form of algorithmic sufficient statistic is also a probabilistic sufficient statistic.

Recall the terminology of probabilistic mutual information (I.1) and probabilistic sufficient statistic (I.2). Consider a probabilistic ensemble of models, a family of computable probability mass functions $\{f_\theta\}$ indexed by a discrete parameter θ, together with a computable distribution

p_1 over θ. (The finite set model case is the restriction where the f_θ's are restricted to uniform distributions with finite supports.) This way we have a random variable Θ with outcomes in $\{f_\theta\}$ and a random variable X with outcomes in the union of domains of f_θ, and $p(\theta, x) = p_1(\theta) f_\theta(x)$ is computable.

Notation VI.1: To compare the algorithmic sufficient statistic with the probabilistic sufficient statistic it is convenient to denote the sufficient statistic as a function $S(\cdot)$ of the data in both cases. Let a statistic $S(x)$ of data x be the more general form of probability distribution as in Section V. That is, S maps the data x to the parameter ρ that determines a probability mass function f_ρ (possibly not an element of $\{f_\theta\}$). Note that "$f_\rho(\cdot)$" corresponds to "$P(\cdot)$" in Section V. If f_ρ is computable, then this can be the Turing machine T_ρ that computes f_ρ. Hence, in the current section, "$S(x)$" denotes a probability distribution, say f_ρ, and "$f_\rho(x)$" is the probability f_ρ concentrates on data x.

Lemma VI.2: Let $p(\theta, x) = p_1(\theta) f_\theta(x)$ be a computable joint probability mass function, and let S be a function. Then all three conditions below are equivalent and imply each other:

(i) S is a probabilistic sufficient statistic (in the form $I(\Theta, X) \stackrel{+}{=} I(\Theta, S(X))$).

(ii) S satisfies

$$\sum_{\theta,x} p(\theta, x) I(\theta : x) \stackrel{+}{=} \sum_{\theta,x} p(\theta, x) I(\theta : S(x)) \tag{VI.1}$$

(iii) S satisfies

$$I(\Theta; X) \stackrel{+}{=} I(\Theta; S(X)) \stackrel{+}{=} \sum_{\theta,x} p(\theta, x) I(\theta : x) \stackrel{+}{=} \sum_{\theta,x} p(\theta, x) I(\theta : S(x)).$$

All $\stackrel{+}{=}$ signs hold up to an $\stackrel{+}{=} \pm 2K(p)$ constant additive term.

Proof: Clearly, (iii) implies (i) and (ii).

We show that both (i) implies (iii) and (ii) implies (iii): By (II.4) we have

$$I(\Theta; X) \stackrel{+}{=} \sum_{\theta,x} p(\theta, x) I(\theta : x), \tag{VI.2}$$
$$I(\Theta; S(X)) \stackrel{+}{=} \sum_{\theta,x} p(\theta, x) I(\theta : S(x)),$$

where we absorb a $\pm 2K(p)$ additive term in the $\stackrel{+}{=}$ sign. Together with (VI.1), (VI.2) implies

$$I(\Theta; X) \stackrel{+}{=} I(\Theta; S(X)); \tag{VI.3}$$

and *vice versa* (VI.3) together with (VI.2) implies (VI.1). ■

Remark VI.3: It may be worth stressing that S in Theorem VI.2 can be any function, without restriction. ♢

Remark VI.4: Note that (VI.3) involves equality $\stackrel{+}{=}$ rather than precise equality as in the definition of the probabilistic sufficient statistic (I.2). ♢

Definition VI.5: Assume the terminology and notation above. A statistic S for data x is *θ-sufficient with deficiency δ* if $I(\theta, x) \stackrel{+}{=} I(\theta, S(x)) + \delta$. If $\delta \stackrel{+}{=} 0$ then $S(x)$ is simply a *θ-sufficient statistic*.

The following lemma shows that θ-sufficiency is a type of conditional sufficiency:

Lemma VI.6: Let $S(x)$ be a sufficient statistic for x. Then,

$$K(x \mid \theta^*) + \delta \stackrel{+}{=} K(S(x) \mid \theta^*) - \log S(x). \tag{VI.4}$$

iff $I(\theta, x) \stackrel{+}{=} I(\theta, S(x)) + \delta$.

Proof: (If) By assumption, $K(S(x)) - K(S(x) \mid \theta^*) + \delta \stackrel{+}{=} K(x) - K(x \mid \theta^*)$. Rearrange and add $-K(x \mid S(x)^*) - \log S(x) \stackrel{+}{=} 0$ (by typicality) to the right-hand side to obtain $K(x \mid \theta^*) + K(S(x)) \stackrel{+}{=} K(S(x) \mid \theta^*) + K(x) - K(x \mid S(x)^*) - \log S(x) - \delta$. Substitute according to $K(x) \stackrel{+}{=} K(S(x)) + K(x \mid S(x)^*)$ (by sufficiency) in the right-hand side, and subsequently subtract $K(S(x))$ from both sides, to obtain (VI.4).

(Only If) Reverse the proof of the (If) case. ■

The following theorems state that $S(X)$ is a probabilistic sufficient statistic iff $S(x)$ is an algorithmic θ-sufficient statistic, up to small deficiency, with high probability.

Theorem VI.7: Let $p(\theta, x) = p_1(\theta) f_\theta(x)$ be a computable joint probability mass function, and let S be a function. If S is a recursive probabilistic sufficient statistic, then S is a θ-sufficient statistic with deficiency $O(k)$, with p-probability at least $1 - \frac{1}{k}$.

Proof: If S is a probabilistic sufficient statistic, then, by Lemma VI.2, equality of p-expectations (VI.1) holds. However, it is still consistent with this to have large positive and negative differences $I(\theta : x) - I(\theta : S(x))$ for different (θ, x) arguments, such that these differences cancel each other. This problem is resolved by appeal to the algorithmic mutual information non-increase law (II.6) which shows that all differences are essentially positive: $I(\theta : x) - I(\theta : S(x)) \stackrel{+}{>} -K(S)$. Altogether, let c_1, c_2 be least positive constants such that $I(\theta : x) - I(\theta : S(x)) + c_1$ is always nonnegative and its p-expectation is c_2. Then, by Markov's inequality,

$$p(I(\theta : x) - I(\theta : S(x)) \geq k c_2 - c_1) \leq \frac{1}{k},$$

that is,

$$p(I(\theta : x) - I(\theta : S(x)) < k c_2 - c_1) > 1 - \frac{1}{k}.$$

■

Theorem VI.8: For each n, consider the set of data x of length n. Let $p(\theta, x) = p_1(\theta) f_\theta(x)$ be a computable joint probability mass function, and let S be a function. If S is an algorithmic θ-sufficient statistic for x, with p-probability at least $1 - \epsilon$ ($1/\epsilon \stackrel{+}{=} n + 2 \log n$), then S is a probabilistic sufficient statistic.

Proof: By assumption, using Definition VI.5, there is a positive constant c_1, such that,

$$p(|I(\theta : x) - I(\theta : S(x))| \leq c_1) \geq 1 - \epsilon.$$

Therefore,

$$0 \leq \sum_{|I(\theta:x)-I(\theta:S(x))|\leq c_1} p(\theta,x)|I(\theta:x)-I(\theta:S(x))| \stackrel{+}{<} (1-\epsilon)c_1 \stackrel{+}{=} 0.$$

On the other hand, since

$$1/\epsilon \stackrel{+}{>} n+2\log n \stackrel{+}{>} K(x) \stackrel{+}{>} \max_{\theta,x} I(\theta;x),$$

we obtain

$$0 \leq \sum_{|I(\theta:x)-I(\theta:S(x))|>c_1} p(\theta,x)|I(\theta:x)-I(\theta:S(x))| \stackrel{+}{<} \epsilon(n+2\log n) \stackrel{+}{<} 0.$$

Altogether, this implies (VI.1), and by Lemma VI.2, the theorem. ■

VII. Conclusion

An algorithmic sufficient statistic is an individual finite set (or probability distribution) for which a given individual sequence is a typical member. The theory is formulated in Kolmogorov's absolute notion of the quantity of information in an individual object. This is a notion analogous to, and in some sense sharper than the probabilistic notion of sufficient statistic—an average notion based on the entropies of random variables. It turned out, that for every sequence x we can determine the complexity range of possible algorithmic sufficient statistics, and, in particular, exhibit a algorithmic minimal sufficient statistic. The manner in which the statistic is effectively represented is crucial: we distinguish implicit representation and explicit representation. The latter is essentially a list of the elements of a finite set or a table of the probability density function; the former is less explicit than a list or table but more explicit than just recursive enumeration or approximation in the limit. The algorithmic minimal sufficient statistic can be considerably more complex depending on whether we want explicit or implicit representations. We have shown that there are sequences that have no simple explicit algorithmic sufficient statistic: the algorithmic minimal sufficient statistic is essentially the sequence itself. Note that such sequences cannot be random in the sense of having maximal Kolmogorov complexity—in that case already the simple set of all sequences of its length, or the corresponding uniform distribution, is an algorithmic sufficient statistic of almost zero complexity. We demonstrated close relations between the probabilistic notions and the corresponding algorithmic notions: (i) The average algorithmic mutual information is equal to the probabilistic mutual information. (ii) To compare algorithmic sufficient statistic and probabilistic sufficient statistic meaningfully one needs to consider a conditional version of algorithmic sufficient statistic. We defined such a notion and demonstrated that probabilistic sufficient statistic is with high probability an (appropriately conditioned) algorithmic sufficient statistic and vice versa. The most conspicuous theoretical open end is as follows: For explicit descriptions we were only able to guarantee a algorithmic minimal near-sufficient statistic, although the construction can be shown to be minimal sufficient for almost all sequences. One would like to obtain a concrete example of a truly explicit algorithmic minimal sufficient statistic. In the theory of sufficient statistic for models that are probability distributions, in contrast to that of finite set models, one has to deal with an extra additive term $K(-\log P(x) \mid P^*)$. It is not known how far that term can be reduced.

Because the Kolmogorov complexity is not computable, an algorithmic sufficient statistic cannot be computed either. Nonetheless, the analysis gives limits to what is achievable in practice—like in the cases of coding theorems and channel capacities under different noise models in Shannon information theory. The theoretical notion of algorithmic sufficient statistic forms the inspiration to develop applied models that can be viewed as computable approximations. Minimum description length (MDL),[1], is a good example; its relation with the algorithmic minimal sufficient statistic is given in [19]. As in the case of ordinary probabilistic statistic, algorithmic statistic if applied unrestrained cannot give much insight into the *meaning* of the data; in practice one must use background information to determine the appropriate model class first—establishing what meaning the data can have—and only then apply algorithmic statistic to obtain the best model in that class by optimizing its parameters. See Example III.5. Nonetheless, in applications one can sometimes still unrestrictedly use compression properties for model selection, for example by a judicious choice of model parameter to optimize. One example is the precision at which we represent the other parameters: too high precision causes accidental noise to be modeled as well, too low precision may cause models that should be distinct to be confusing. In general, the performance of a model for a given data sample depends critically on what we may call the "degree of discretization" or the "granularity" of the model: the choice of precision of the parameters, the number of nodes in the hidden layer of a neural network, and so on. The granularity is often determined ad hoc. In [8], in two quite different experimental settings the best model granularity values predicted by MDL are shown to coincide with the best values found experimentally.

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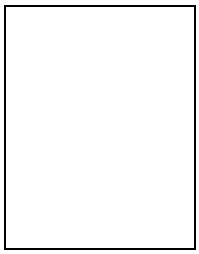

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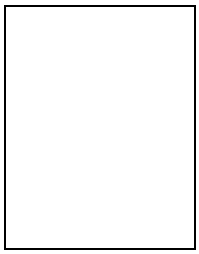

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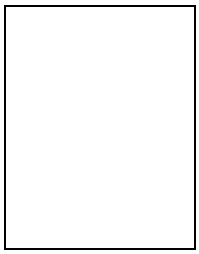

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