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# ON THE SYMMETRY OF ALGORITHMIC INFORMATION

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P. GAČ

The complexity $K(x)$ of the algorithmic definition of a finite binary word $x$ was introduced by A. N. Kolmogorov in [1]. Kolmogorov and Levin showed that the information $I(x:y) = K(y) - K(y/x)$ concerning the word $y$ in the word $x$, though it is symmetric to a greater degree (to within the logarithm of complexity), it is not exactly (i.e. not to within an additive constant) a symmetric function of its arguments (for a proof see [3]). There exist several variants of complexity, introduced by Levin in [4], [5], which asymptotically coincide with it and yield probabilistic results that are simpler in form. Symmetry for them to within the logarithm of complexity at once follows from the results mentioned above, but the degree of nonsymmetry is not known. One of these complexities, $KP(x)$, considered in detail by Levin, is defined as the complexity in specifying $x$ on a machine on which it is impossible to indicate the halting of a master program: an infinite sequence of binary symbols enters the machine and the machine must itself decide how many binary symbols are required for its computation. $KP(x)$, on the other hand, is equal to the base two logarithm of a universal semicomputable probability measure that can be defined on the countable set of all words.

The aim of the present article is to give exact relationships of nonsymmetry for $IP(x:y) = KP(y) - KP(y/x)$. Nonsymmetry of a similar order follows from them and for other complexities as well, since they coincide with $KP(x)$ on any prefix set. In order to maintain unity of presentation and not to complicate the article with references, we announce at the outset: the relationship

$$KP(x,\ KP(x)) \asymp KP(x) \tag{1}$$

is due to the author, as well as Theorem 2 and its Corollary to the effect that $IP(x: y)$ is nonsymmetric. All the other results and corollaries were obtained by Levin.

(1) is a unique, nonobvious property which is required of a certain complexity function $\widetilde{K}(x)$ in order to prove the nonsymmetry of $\widetilde{I}(x:y)$. It is quite natural to substantiate this in a certain axiomatics considered by Levin, which, moreover, in a natural way distinguishes $KP(x)$ from other conceivable complexities.

The last theorem shows that, although $IP(x:y)$ is nonsymmetric to the same order as is $I(x:y)$, nonsymmetry for $IP(x:x)$ is an extremely rare event, while for $I(x:y)$ greater nonsymmetricity is reached on random sequences.

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Let us first give a precise relationship of nonsymmetry for $IP(x:y)$.

**Theorem 1.**

$$KP(x)+KP(y/x,\ KP(x)) \asymp KP(x,\ y). \tag{2}$$

**Proof.** 1) ($\succcurlyeq$). If we have the shortest program that assigns $x$, then as a result of determining $KP(x)$ the machine will know both its length and its end, and it can write down the shortest program assigning $y$ for $x$ and $KP(x)$.

2) ($\preccurlyeq$).

$$KP(y/x,\ KP(x)) \preccurlyeq KP(x,\ y)-KP(x), \tag{3}$$

since $2^{KP(x)-KP(x,y)}$ is a probability distribution (to within a multiplicative constant) on the words of $y$ which is semicomputable relative to $x$ and $KP(x)$.

**Corollary 1.**

$$KP(x,\ y) \asymp KP(x,\ KP(x),\ y). \tag{4}$$

**Corollary 2.**

$$\begin{aligned} &KP(x)+KP(y/x)-KP(x,\ y) \asymp IP(KP(x):y/x) \\ &\preccurlyeq KP(KP(x)/x) \preccurlyeq \log_2 KP(x)+2\log_2\log_2 KP(x). \end{aligned} \tag{5}$$

**Corollary 3.**

$$IP(x:y)-IP(y:x) \asymp IP(KP(y):x/y)-IP(KP(x):y/x). \tag{6}$$

We will now show that on some infinite set, the left side in (5) and (6) is asymptotically equal to $\log_2 KP(x,\ y)$. Let $\tilde{x} = \langle x,\ KP(x)\rangle$. Then

$$KP(x)+KP(\tilde{x}/x)-KP(x,\ \tilde{x}) \asymp KP(KP(x)/x), \tag{7}$$

$$IP(x:\tilde{x})-IP(\tilde{x}:x) \asymp -KP(KP(x)/x). \tag{8}$$

If $l(x)=n$, then $KP(KP(x)/x) \preccurlyeq \log_2 n$. Therefore our assertion reduces to the following theorem.

**Theorem 2.** *For all $n$ there exists $x$ of length $n$ such that*

$$KP(KP(x)/x) \succcurlyeq \log_2 n-\log_2\log_2 n. \tag{9}$$

**Proof.** Let $B(p,\ x)$ be that universal partial recursive function for which $KP(y/x)=\min_{B(p,x)=y} l(p)$. Let $D=\{0,\ 1\}$.

Let $s \leq \log_2 n$ be such that if $l(x)=n$, then $P \in D^s$ can be found for which $B(p,\ x)=KP(x)$. We must show that

$$s \succcurlyeq \log_2 n-\log_2\log_2 n. \tag{10}$$

Let us say that $p \in D^s$ is "suitable" for $x \in D^n$ if there exists $k=B(p,\ x)$ and $q \in D^k$ such that $B(q,\ \Lambda)=x$.

Let us denote by $M_i$ the set of those $x \in D^n$ for which there exist at least $i$ different suitable $p \in D^s$. We will consider the sequence $D^n = M_0 \supset M_1 \supset \cdots \supset M_j \supset M_{j+1} \neq \emptyset$, where $M_j \neq \emptyset$. It is clear that $2^s \geq j$. We now show that

$$\log_2 |M_i| \preccurlyeq \log_2 |M_{i+1}| + 4 \log n. \tag{11}$$

In fact, we may write a program with a length given by the second term of the inequality (11), which assigns $x \in M_i - M_{i+1}$ with the property $KP(x) \geq \log |M_i| - 1$. The program operates in the following manner. It assigns $|M_{i+1}|$ by means of a segment of length $\log_2 |M_{i+1}| + \log_2 n + 2 \log_2 \log_2 n$. The numbers $i$, $n$, and $s$ are assigned by means of programs of lengths $\log_2 n + 2 \log_2 \log_2 n$, $n + 2 \log \log n$, and $2 \log_2 \log_2 n$, respectively. From this data the machine can determine the set $M_{i+1}$ and then begin to enumerate the set $M_i - M_{i+1}$. For $x \in M_i - M_{i+1}$ it can determine $KP(x)$. Since we may assume that $\log_2 |M_i - M_{i+1}| \geq \log_2 |M_i| - 1$ (otherwise (11) is trivial), there exists $x \in M_i - M_{i+1}$ for which $KP(x) \geq \log_2 |M_i| - 1$. The machine delivers the first such $x$ as the result.

From the inequality (11) it immediately follows that $j \geq n/(4 \log_2 n)$, and (10) follows from the latter.

**Corollary.** *$IP(x:y)$ (as well as $I(x:y)$) is not even asymptotically symmetric.*

**Proof.** Let us assume that $l(x) = n$ and $KP(KP(x)/x) \succcurlyeq \log_2 n - \log_2 \log_2 n$. We will consider $x$, $\tilde{x}$, and $KP(x)$. The following relationships are immediately evident:

$$IP(x:KP(x)) \preccurlyeq 3 \log_2 \log_2 n < \log_2 n - \log_2 \log_2 n \preccurlyeq IP(\tilde{x}:KP(x)), \tag{12}$$

while

$$IP(KP(x):x) \asymp IP(KP(x):\tilde{x}). \tag{13}$$

It follows from this that symmetry breaks down in an exponentially greater measure either on the pair $x$, $K(x)$ or on the pair $x$, $\tilde{K}(x)$.

The following theorem, which we give without proof, shows that the words $x$, $y$ on which $IP(x, y)$ is nonsymmetric are rather "exotic".

Information $IP(\alpha: x)$ in an infinite sequence $\alpha$ concerning the word $x$ is determined in a natural manner.

**Theorem 3.** *Let $\nu$ be the characteristic function of a universal recursively enumerable set. Then*

$$KP(KP(x)/x) \preccurlyeq IP(\nu:x) + 3 \log_2 IP(\nu:x). \tag{14}$$

Thus if $KP(KP(x)/x)$ is large, much information concerning $x$ is contained in the universal set. We note that for any computable or semicomputable probability distribution $k$ units of information concerning $\nu$ appear in $x$ with a probability not greater than $2^{-k}$.

Moscow State University Received 9/APR/74

BIBLIOGRAPHY

1. A. N. Kolmorogov, *Three approaches to the definition of the concept of "amount of information"*, Problemy Peredači Informacii 1 (1965), no. 1, 3–11; English transl., Selected Transl. Math. Statist. and Probability, vol. 7, Amer. Math. Soc., Providence, R. I., 1968, pp. 293–302. MR 32 #2273.

2. ———, *Some theorems on algorithmic entropy and algorithmic quantity of information,* Uspehi Mat. Nauk 23 (1968), no. 2 (140), 201. (Russian)

3. A. K. Zvonkin and L. A. Levin, *Complexity of finite objects and the algorithm-theoretic foundations of the notions of information and randomness,* Uspehi Mat. Nauk **25** (1970), no. 6 (156), 85–127 = Russian Math. Surveys 25 (1970), no. 6, 83–124.

4. L. A. Levin, *On the notion of a random sequence*, Dokl. Akad. Nauk SSSR **212** (1973), 548–550 = Soviet Math. Dokl. **14** (1973), 1413–1416.

5. ———, *Laws of information conservation nonincrease and problems of the foundation of probability theory*, Problemy Peredači Informacii 10 (1974), no. 3, 30–35 = Problems of Information Transmission **10** (1974) (to appear).


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