Solution for HW4

Problem 1:
$a.b = a.c \Rightarrow b.a.b = b.a.c$ (we know $b.a = 1$) $\Rightarrow b = c$

Problem 2:
(i) $a0 = 0$ for all a in R.
$a.b = a.(b + 0)$ (Additive identity)
$= (a.b) + (a.0)$ (Distributivity of multiplication over addition)
$\Rightarrow a.0 = 0$

(ii)$(-a)(-b) = ab$ for all a, b in R.
$(-a).(b + -b) = (-a).(b) + (-a).(-b) = 0$
$\Rightarrow (-a).(b)$ is negetive of $(-a).(-b)$
$\Rightarrow (-a).(-b) = -(-a).b = a.b$ (we proved that $-(-a) = a$)

(iii) $(na)b = a(nb) = n(ab)$ for all n in N, and a, b in R.
$(na)b = \underbrace{((a + a + ... + a) + a)}_{n} b = \underbrace{(a + ... + a)}_{n-1} b \ + \ ab \ = ... = \underbrace{ab + ... + ab}_{n} = n(ab)$
Same method is applied for $a(nb) = n(ab)$

Problem 3:
$a \neq 0$, If $ar = as \Rightarrow r = s$
Since a is a nonzero element of field F, a has an inverse:
$ar = as \Rightarrow a^{-1}ar = a^{-1}as \Rightarrow r = s.$

Problem 4:
All axioms hold for N except negetive element.

Problem 5:
m is odd, so $m = 2n + 1$
$[2]_m.[x]_m \equiv [1]_m \Rightarrow 2x \equiv \ 1(mod\ m) \Rightarrow 2x = m + 1 = 2n + 2 \Rightarrow x = n + 1$

Problem 6:
$\{[1]_{21}, [2]_{21}, [4]_{21}, [5]_{21}, [8]_{21}, [10]_{21}, [11]_{21}, [13]_{21}, [16]_{21}, [17]_{21}, [19]_{21}, [20]_{21}\}$
For all a, b in the set of all units, since $(a, m) = 1$ and $(b, m) = 1$ so $(ab, m) = 1 \Rightarrow ab$ is a unit.

Problem 7:
Since S is a subring of R, so if S has some zero divisors, then they are zero divisors for R too, while we know R has no zero divisor.

Problem 8:
Since a is a zero divisor in R, so $\exists b \in R, b \neq 0$ s.t $ab = 0$. So $x = \{0, b\}$

Problem 9:
$[a]_m$ is a zero divisor, so $(a, m) > 1 \Rightarrow a$ is not a unit. So a has no inverse.

Problem 10:
$[2]_{16} : [8]_{16}$
$[4]_{16} : [4]_{16}, [8]_{16}, [12]_{16}$
$[6]_{16} : [8]_{16}$
$[8]_{16} : [2]_{16}, [4]_{16}, [6]_{16}, [10]_{16}, [12]_{16}, [14]_{16}$
$[10]_{16} : [8]_{16}$
$[14]_{16} : [8]_{16}$