Solutions for AS1

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Problem 1
We have a | b so b = ja, for some j.
We have a | c so c = ka, for some k.
Let d = (b,c), then by Bezout's Lemma, there are integers x and y with d = bx + cy.
So substituting from above, d = bx + cy = jax + kay = a(jx+ky).
Hence, since a | a(jx+ky), a | d as desired.

Problem 2
False.
For example, $a = 10, b = 4, c = 5$.

Problem 3
Base case: $n = 1$, $1 + 2^1 = 3$, and $2^{1+1} - 1 = 3$.
Assume for some $k > 1$, the claim is true, and we have $1 + 2^1 + 2^2 + ... + 2^k = 2^{k+1} - 1$.
Then for $k + 1$, we have $1 + 2^1 + 2^2 + ... + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} = 2^{k+2} - 1$. The claim is proven.

Problem 4
If $a|b$, $\exists c$, $b = ca$. $(a, b) = (a, ca) = a(1, c) = a$.

Problem 5
gcd of $2^7 3^2 4^5 5^6 6^5$ and $2^4 3^5 4^3 5^3 6^7$
$(2^7 3^2 4^5 5^6 6^5, 2^4 3^5 4^3 5^3 6^7) = 2^4 3^2 4^3 5^3 6^5 (2^3 4^2 5^3, 3^3 6^2) = 2^4 3^2 4^3 5^3 6^5 (2^7 5^3, 3^5 2^2) = 2^6 3^2 4^3 5^3 6^5 (2^5 5^3, 3^5) = 2^6 3^2 4^3 5^3 6^5$

Problem 6
Let $a = cp^3$, $b = dp^3$, from Bezout's Lemma, $\exists j, k \in \mathbb{Z}$, such that
$p^3 = aj + bk = cp^3 j + dp^3 k$, $1 = cj + dk$, it is equivalent to 1 = (c,d).
We can write $c = p_1^{k_1} ... p_n^{k_n}$, and $d = p_1'^{k_1'} ... p_m'^{k_m'}$, where $p_1, ...p_n$ are different primes from $p_1', ..., p_m'$.
$c^2 = p_1^{2k_1} ... p_n^{2k_n}$, and $d^2 = p_1'^{2k_1'} ... p_m'^{2k_m'}$. We can see that $(c^2, d^2) = 1$.
$(a^2, b^2) = (c^2 p^6, d^2 p^6) = p^6 (c^2, d^2) = p^6$.

Problem 7
As long as $(a, b) = p_1$, $(b, c) = p_2$, $(a, c) = p_3$, and p_1, p_2 and p_3 are different primes, let $a = p_1 p_3$, $b = p_1 p_2$ and $c = p_2 p_3$, then $(a, b, c) = 1$. It is easy to prove it, $(a, b, c) = ((a, b), c) = (p_1, p_2 p_3) = 1$.

Problem 8
$(a, m) = d$, $(b, m) = 1$, let $a = dk_1, m = dk_2$, then we have $(a, m) = (dk_1, dk_2) = d(k_1, k_2) = d$, and $(k_1, k_2) = 1$.
$(b, m) = 1$, from Bezout's Lemma, we have $\exists j, k \in \mathbb{Z}$, such that
$1 = bj + mk = bj + k_2dk$, so $(b, k_2) = 1$.
If we represent b, k_1, k_2 in prime factors form, b and k_1 have different prime factors from k_2, so we have $(bk_1, k_2) = 1$.
$(ab, m) = (dk_1b, dk_2) = d(k_1b, k_2) = d$.