CS 630 - Fall 2021
Homework 3

Due: Friday, November 12 by 9:00pm - submit via Gradescope

Reading : Read Chapter 35, sections 1, 2, amd 3, 1106-1122 and Chapter 34, the introduction and sections 2 and 3, pages 1061 to 1070. (Look over section 1.)

Problems:

1. The Randomized max-cut problem is to find an efficient randomized algorithm which takes as input a graph G and which outputs a cut (S,T) of G. In this problem we also want to compare this algorithm to the 2-approximation for maxcut given in class.

Here is the randomized algorithm for constructing a cut C = (S,T) in a graph G:

Go through the list of vertices $v_1, v_2, v_3, ..., v_n$ in G.
For each vertex v_i flip a fair coin and if it comes up heads put the vertex in S, otherwise put it in T. (As I said, it is pretty simple).

Answer questions i., ii. and iii. below about this randoized algorithm:

(i). What is the expected value of $|C| = |(S,T)|$ constructed above ?

Your answer should show the expected value of the cut $|C|$, in terms of the number m of edges in G. You should briefly explain how you calculated this expected value and what properties of expected value you used to obtain your answer.

(ii). Compare this algorithm to the 2-approximation for maxcut given in class.
Specifically, does it give stronger result or a weaker result (or an incomparable) result to the deterministic algorithm given in class? Briefly explain why you think that.

2. This problem is the same as problem 2, only for a weighted graph.
You still use the same algorithm as given before part (i) in problem 2 above. The weight of the cut you get here is the sum of the weights of the edges across the cut.

What is the expected value of the weighted cut (S,T) constructed above ?
This question is pretty much the same as before as well, but now the size of the cut is the sum of the weights of the edges crossing the cut.
Your answer should give the expected value in terms of the weights of the edges of G.
Again you should show how you calculated this expected value.

Do you think this gives us a 2-approximation for the weightd max cut problem ? Explain why or why not ?

3. Give an example of a connected graph G where the algorithm described on page 1111, Problem 35.1-3 is used and which outputs a VC which is ≥ 1.5 times the size of the optimal VC. Note: This is the algorithm which always selects the vertex of largest degree remaining in the graph and puts that vertex into the VC you are constucting.

(i). State the smallest VC in your graph G that you can find and write the vertex cover that your approximation algorithm above gives you in this case.

(ii). Show how the algrithm can work on G and gives a vertx cover C satisfying whose size is more than 1.5 times the size of the best vertex cover.

4. Refer to the algorithm for the TSP problem in our textbook which yields a TSP cycle which is ≤ 2 times the size of the smallest TSP cycle. Recall his algorithm works when the graph satisfies the triangle inequality.

i. Give an example of a complete graph F which satisfies the triangle inequality and where this approximation is more than 1.5 times the the size of the optimal (smallest) TSP cycle in F.

iii. Show the steps of the algorithm (as given on page 1114 Theorem 35.2) on your graph F and state which MSWT your algorithm chooses as well as the size of the TSP cycle your algorithm finds.

5. Consider the randomized min-cut algorithm (Karger's algorithm) which comes from the randomized algorithm notes posted on the home page.

Suppose that we modify our algorithm as follows: At each step of our min-cut algorithm, instead of choosing a random edge for contraction we choose two vertices at random and contract those vertices together (even if there is no edge between them). We still halt after $|V|-2$ contraction steps

Give an example of a graph where, when we use the modified min-cut algorithm the probability of find a min-cut is exponentially small. (That is the probability should be no more than 1/f(n) where f(n) is some exponential function.) Give a proof of this if you can, or if not, do your best to explain your reaoning.