

# Generalized $\lambda$ -calculi

## (Abstract)

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We propose a notion of generalized  $\lambda$ -calculi, which include the usual call-by-name  $\lambda$ -calculus, the usual call-by-value  $\lambda$ -calculus, and many other  $\lambda$ -calculi such as the  $\lambda_g$ -calculus[3], the  $\lambda_{hd}^v$ -calculus[5], etc. We prove the Church-Rosser theorem and the standardization theorem for these generalized  $\lambda$ -calculi. The normalization theorem then follows, which enables us to define evaluation functions for the generalized  $\lambda$ -calculi. Our proof technique mainly establishes on the notion of *separating developments*[4], yielding intuitive and clean inductive proofs.

This work aims at providing a solid foundation for *evaluation under  $\lambda$ -abstraction*, a notion which is pervasive in both partial evaluation and run-time code generation for functional programming languages.

**Definition 1.** We use the following for  $\lambda$ -terms and contexts:

$$\text{(terms)} \quad L, M, N ::= x \mid (\lambda x.M) \mid M(N) \quad \text{(contexts)} \quad C ::= [] \mid (\lambda x.C) \mid M(C) \mid C(M)$$

We use  $FV(M)$  for the set of free variables in  $M$ .

**Definition 2.** (General  $\lambda$ -abstraction) We define function  $\text{abs}$  on  $\lambda$ -terms as follows:

$$\text{abs}(x) = 0 \quad \text{abs}(\lambda x.M) = \text{abs}(M) + 1 \quad \text{abs}(M(N)) = \text{abs}(M) \div 1$$

Note  $n \div 1 = n - 1$  if  $n > 0$  and  $0 \div 1 = 0$ .  $M$  is a general  $\lambda$ -abstraction if  $\text{abs}(M) > 0$ .

We use  $A$  for the set of  $\lambda$ -terms; **lam** for the set of  $\lambda$ -abstractions; **glam** for the set of general  $\lambda$ -abstractions; **var** for the set of variables.

**Definition 3.** The body of a general  $\lambda$ -abstraction  $M$  is defined as  $\text{bd}(M) = \text{gbd}(M, 0)$ , where  $\text{gbd}$  is defined as follows.

$$\text{gbd}(\lambda x.M, 0) = M[x := \bullet] \quad \text{gbd}(\lambda x.M, n+1) = \lambda x.\text{gbd}(M, n) \quad \text{gbd}(M(N), n) = \text{gbd}(M, n+1)(N)$$

A general redex is of form  $M(N)$  where  $M$  is a general  $\lambda$ -abstraction. The contractum of a general redex  $M(N)$  is  $\beta(M, N) = \text{bd}(M)[\bullet := N]$ .

**Definition 4.** Let  $\mathcal{S}_1$  and  $\mathcal{S}_2$  be sets of  $\lambda$ -terms; we say  $\mathcal{S}_1$  is closed under  $\mathcal{S}_2$  if  $M[x := N] \in \mathcal{S}_1$  for all  $x \in FV(M)$  and  $N \in \mathcal{S}_2$ .  $\mathcal{R} = \langle \mathcal{F}, \mathcal{V} \rangle$  is a closed redex set(c.r.s.) if  $\mathcal{F}$  contains only general  $\lambda$ -abstractions and both  $\mathcal{F}$  and  $\mathcal{V}$  are closed under  $\mathcal{V}$ .

**Definition 5.** Given a closed redex set  $\mathcal{R} = \langle \mathcal{F}, \mathcal{V} \rangle$ ;  $M(N)$  is a  $\beta_{\mathcal{R}}$ -redex if  $M \in \mathcal{F}$  and  $N \in \mathcal{V}$ ;  $M_1 \xrightarrow{\beta}_{\mathcal{R}} M_2$  if  $M_1 = C[M(N)]$  for some  $\beta_{\mathcal{R}}$ -redex  $M(N)$  and  $M_2 = C[\beta(M, N)]$ ;  $\xrightarrow{\beta}_{\mathcal{R}}$  is the reflexive and transitive closure of  $\xrightarrow{\beta}_{\mathcal{R}}$ ; we use  $\sigma$  for a (finite)  $\beta_{\mathcal{R}}$ -reduction sequence, and  $\sigma(M)$  for the  $\lambda$ -term to which  $\sigma$  reduces  $M$ .

Given a c.r.s.  $\mathcal{R}$ ; the general  $\lambda$ -calculus  $\lambda_{\mathcal{R}}$  studies the reduction  $\xrightarrow{\beta}_{\mathcal{R}}$ . We write  $\lambda_{\mathcal{R}} \vdash M \equiv_{\mathcal{R}} N$  if there exist  $M = M_0, M_1, \dots, M_{2n-2}, M_{2n} = N$  such that  $M_{2i+1} \xrightarrow{\beta}_{\mathcal{R}} M_{2i}$  and  $M_{2i+1} \xrightarrow{\beta}_{\mathcal{R}} M_{2i+2}$  for  $0 \leq i < n$ .

*Remark.* The (usual call-by-name)  $\lambda$ -calculus is  $\lambda_{\mathcal{R}}$  for  $\mathcal{R} = \langle \mathbf{lam}, A \rangle$ ; the (usual) call-by-value  $\lambda$ -calculus is  $\lambda_{\mathcal{R}}$  for  $\mathcal{R} = \langle \mathbf{lam}, \mathbf{lam} \cup \mathbf{var} \rangle$ ; the  $\lambda_g$  in [3] is  $\lambda_{\mathcal{R}}$  for  $\mathcal{R} = \langle \mathbf{glam}, A \rangle$ ; the  $\lambda_{hd}^v$  in [5] is  $\lambda_{\mathcal{R}}$  for  $\mathcal{R} = \langle \mathbf{ghnf}, \mathbf{ghnf} \rangle$ , where **ghnf** is the set of  $\lambda$ -terms in generalized head normal form[5]; the call-by-need  $\lambda$ -calculus[1] closely relates to  $\lambda_{\mathcal{R}}$  for  $\mathcal{R} = \langle \mathbf{ghnf}, \mathbf{lam} \cup \mathbf{var} \rangle$ . It can be readily verified that every  $\mathcal{R}$  mentioned above is a c.r.s.

The notion of *residuals* of a  $\beta_{\mathcal{R}}$ -redex under  $\beta_{\mathcal{R}}$ -reductions can be defined as usual[2]. Note that the conditions imposed on the definition of closed redex set are crucial for making the definition go through.

**Definition 6.** (Involvedness) Given a  $\beta_{\mathcal{R}}$ -reduction sequence  $\sigma$  from  $M$ ; a  $\beta_{\mathcal{R}}$ -redex in  $M$  is involved in  $\sigma$  if the  $\beta_{\mathcal{R}}$ -redex or one of its residuals is contracted in  $\sigma$ .

**Definition 7.** ( $\beta_{\mathcal{R}}$ -development) Given a  $\lambda$ -term  $M$  and a set  $\mathcal{S}$  of  $\beta_{\mathcal{R}}$ -redex in  $M$ ;  $\sigma : M \xrightarrow{\beta}_{\mathcal{R}} N$  is a  $\beta_{\mathcal{R}}$ -development (of  $\mathcal{S}$ ) if it contracts only  $\beta_{\mathcal{R}}$ -redexes in  $\mathcal{S}$  and their residuals.

**Lemma 8.** (Separation) Let  $M = M_1(M_2)$  be a  $\beta_{\mathcal{R}}$ -redex and  $\sigma$  be a  $\beta_{\mathcal{R}}$ -development  $\sigma$  from  $M$  in which  $M$  is involved;  $\sigma(M)$  is of form

$$\sigma_1(\text{bd}(M_1))[\sigma_{21}(M_2), \dots, \sigma_{2n}(M_2)],$$

where  $\sigma_1$  is a  $\beta_{\mathcal{R}}$ -development from  $\text{bd}(M_1)$  and  $\sigma_{2i}$  are  $\beta_{\mathcal{R}}$ -developments from  $M_2$  for  $i = 1, \dots, n$ .

Lemma 8 plays a major rôle in the proofs of the following theorems. Please see [4] for details.

**Theorem 9.** (Church-Rosser) For any given c.r.s.  $\mathcal{R}$ , if  $\lambda_{\mathcal{R}} \vdash M_1 \equiv_{\mathcal{R}} M_2$ , then there exists  $N$  such that  $M_i \xrightarrow{\beta}_{\mathcal{R}} N$  for  $i = 1, 2$ .

**Definition 10.** Let  $\mathcal{R} = \langle \mathcal{F}, \mathcal{V} \rangle$  be a c.r.s. and  $\beta_{\mathcal{R}}(M)$  be the set of all  $\beta_{\mathcal{R}}$ -redexes in  $M$  for every  $\lambda$ -term  $M$ ; a relation on  $\beta_{\mathcal{R}}(M)$  is given as follows.

$$\begin{aligned} \triangleleft_{\mathcal{R}}(M) &= \emptyset && \text{if } M \text{ is a variable;} \\ \triangleleft_{\mathcal{R}}(\lambda x.M) &= \triangleleft_{\mathcal{R}}(M) && ; \\ \triangleleft_{\mathcal{R}}(M(N)) &= \triangleleft_{\mathcal{R}}(M) \cup \triangleleft_{\mathcal{R}}(N) \cup (\beta_{\mathcal{R}}(N) \times \beta_{\mathcal{R}}(M)) \cup \{(M(N), L) : L \in \beta_{\mathcal{R}}(M) \cup \beta_{\mathcal{R}}(N)\} && \text{if } M(N) \text{ is a } \beta_{\mathcal{R}}\text{-redex;} \\ \triangleleft_{\mathcal{R}}(M(N)) &= \triangleleft_{\mathcal{R}}(M) \cup \triangleleft_{\mathcal{R}}(N) \cup (\beta_{\mathcal{R}}(N) \times \beta_{\mathcal{R}}(M)) && \text{if } M \in \mathcal{F} \text{ and } N \notin \mathcal{V}; \\ \triangleleft_{\mathcal{R}}(M(N)) &= \triangleleft_{\mathcal{R}}(M) \cup \triangleleft_{\mathcal{R}}(N) \cup (\beta_{\mathcal{R}}(M) \times \beta_{\mathcal{R}}(N)) && \text{if } M \notin \mathcal{F}. \end{aligned}$$

Note that  $\triangleleft_{\mathcal{R}}(M)$  is a linear order for every  $M$ ; the standard  $\beta_{\mathcal{R}}$  reduction sequences can then be defined accordingly, which leads to the following theorem.

**Theorem 11.** (Standardization) Given any  $\beta_{\mathcal{R}}$ -reduction sequence  $\sigma : M \xrightarrow{\beta}_{\mathcal{R}} N$ ; then there exists a standard  $\beta_{\mathcal{R}}$ -reduction sequence  $\text{std}_{\mathcal{R}}(\sigma) : M \xrightarrow{\beta}_{\mathcal{R}} N$ .

Let the *first*  $\beta_{\mathcal{R}}$ -redex in  $M$  be the first one according to order  $\triangleleft_{\mathcal{R}}(M)$ , then the normalizing strategy is the one which always reduces the first  $\beta_{\mathcal{R}}$ -redex in a term.

**Corollary 12.** (Normalization) If  $\lambda_{\mathcal{R}} \vdash M \equiv_{\mathcal{R}} N$  for some  $N$  in  $\beta_{\mathcal{R}}$ -normal form, then the normalizing strategy reduces  $M$  to  $N$ .

We can then define a evaluation function for  $\lambda_{\mathcal{R}}$  according to the normalizing strategy, establishing a functional programming language upon  $\lambda_{\mathcal{R}}$ .

In conclusion, we have shown that the generalized  $\lambda$ -calculi can unify many existing  $\lambda$ -calculi. We are currently studying  $\lambda_{hd}^v$ , investigating its application to partial-evaluation and run-time code generation.

## References

1. Z.M. Ariola, M. Felleisen, J. Maraist, M. Odersky and P. Wadler (1996), A Call-by-Need Lambda Calculus, *Proceedings of the 22<sup>nd</sup> ACM Symposium on Principles of Programming Languages*, San Francisco, pp. 233-246
2. H.P. Barendregt, *The Lambda Calculus: Its Syntax and Semantics*, North-Holland, Amsterdam, 1984.
3. F. Kamareddine and R. Nederpelt (1996), A useful lambda-notation, *Theoretical Computer Science*, vol. 155(1), pp 85-109.
4. Hongwei Xi (1996), Separating Developments, *Manuscripts*. Available through:  
<http://www.cs.cmu.edu/~huxi/papers/SepDev.ps>
5. Hongwei Xi (1997), Evaluation under lambda-abstraction. In Proceedings of Ninth International Symposium on Programming Languages, Implementations, Logics, and Programs(to appear)
6. N. Yoshida (1993), Optimal reduction in weak  $\lambda$ -calculus with shared environments. In *Proc. ACM conference on Functional Programming Languages and Computer Architecture*, Copenhagen.