CS520 Problems for Assignment 11 (Part II)

Posted: 6 December 2004
Due: 13 December 2004

Throughout this problem set, use sums and sum types according to their presentation in Handout 24, and recursive types according to their presentation in Handout 25.

Problem 5: We want you to show that arithmetic expressions (natural numbers and their prim ops) can be taken as sugar, i.e., defined using other features in the language, in the presence of `Unit`, sum types and recursive types. Specifically, we consider `Nat` no longer as a primitive type, but rather defined by:

$$\mathtt{Nat} \triangleq \mu\mathtt{X}.\mathtt{Unit} + \mathtt{X}$$

Following the interpretation given in lecture, `Nat` can be viewed as the infinite expression:

$$\mathtt{Unit} + \mathtt{Unit} + \mathtt{Unit} + \mathtt{Unit} + \cdots$$

The parse tree of which is:

```
     +
    / \
Unit   +
      / \
  Unit   +
        / \
    Unit   +
          / \
      Unit   ⋮
```

Another way of understanding `Nat` is to take it as the (necessarily infinite) type satisfying the following equation:

$$\mathtt{Nat} = \mathtt{Unit} + \mathtt{Nat}$$

To avoid confusion between natural numbers 0, 1, 2, ... and the terms representing them, we will write $\lceil 0 \rceil$, $\lceil 1 \rceil$, $\lceil 2 \rceil$, ..., for the latter.

(1) Based on the definition of `Nat` as a recursive type above, give an appropriate definition for $\lceil 0 \rceil$ and then an appropriate definition for $\lceil n \rceil$ where n is a natural number > 0.

(2) Based on the definitions in (1), give an appropriate definition for the successor operation `succ`. (Note you must take `succ` as an operation of type `Nat` $\rightarrow$ `Nat`, not as a constructor.)

(3) Based on the definitions in (1), give an appropriate definition for the predecessor operation `pred`. (Assume that (`pred` 0) evaluates to 0.)

(4) Based on the definitions in (1), give an appropriate definition for the test predicate `iszero` of type `Nat` $\to$ `Bool`.

(5) Based on the preceding definitions, verify each of the following:

(a) $(\texttt{succ}\,\lceil n \rceil) \Rightarrow^* \lceil n+1 \rceil$

(b) $(\texttt{iszero}\,\lceil 0 \rceil) \Rightarrow^* \texttt{true}$ and $(\texttt{iszero}\,\lceil n \rceil) \Rightarrow^* \texttt{false}$ for $n > 0$.

(c) $(\texttt{pred}\,(\texttt{succ}\,\lceil x \rceil)) \Rightarrow^* x$ where x is a variable of type `Nat`.

Problem 6: Exercise 20.2.2 in [Pierce, page 278]. Also, can you do the same for the *equi-recursive* approach as presented in [Pierce, page 276]? Answer carefully, pinpointing where there may be complications, if any.