Problem 1.
step 1: We set up the augmented matrix \( AA = [A \ b] \) using Matlab/Octave:

\[
>> \ AA = [2 \ 3 \ 0 \ 1 \ 1 ; \ 2 \ 6 \ 2 \ 2 \ 3 ; \ 2 \ 9 \ 2 \ 1 \ 3]
\]

\[
AA = \\
2 \ 3 \ 0 \ 1 \ 1 \\
2 \ 6 \ 2 \ 2 \ 3 \\
2 \ 9 \ 2 \ 1 \ 3
\]

step 2: using function rref from library, in order to find the row reduced echelon form of \( AA \):

\[
>> \ \text{rref}(A)
\]

\[
\text{ans} = \\
1 \ 0 \ 0 \ 1 \ 1/2 \\
0 \ 1 \ 0 \ -1/3 \ 0 \\
0 \ 0 \ 1 \ 1 \ 1
\]

step 3: The reduced form matrix can be interpreted as:
\[
x_1 + x_4 = 1/2 \rightarrow x_1 = 1/2 - x_4 \\
x_2 - 1/3x_4 = 0 \rightarrow x_2 = 1/3x_4 \\
x_3 + x_4 = 1 \rightarrow x_3 = 1 - x_4
\]

Hence the form of the general solutions for this system of equation is:
\[
X = (x_1, x_2, x_3, x_4) = (1/2 - x_4, 1/3x_4, 1 - x_4, x_4) = (1/2, 0, 1, 0) + x_4(-1, 1/3, -1, 1)
\]
Problem 2.

step 1: We set up the augmented matrix \( AA = [A \ b] \) using Matlab/Octave:

\[
\text{>> } AA = \begin{bmatrix}
-1 & 2 & -1 & 1 \\
-2 & 5 & 1 & 4 \\
-1 & 4 & 5 & 5 \\
-1 & 3 & 2 & 3
\end{bmatrix}
\]

\[
AA = \\
-1 & 2 & -1 & 1 \\
-2 & 5 & 1 & 4 \\
-1 & 4 & 5 & 5 \\
-1 & 3 & 2 & 3
\]

step 2: using function \texttt{rref} from library in order to find the row reduced echelon form of \( AA \):

\[
\text{>> } \texttt{rref}(AA)
\]

\[
\text{ans} = \\
1 & 0 & 7 & 3 \\
0 & 1 & 3 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\]

step 3: The reduced form matrix can be interpreted as:

- \( x_1 + 7x_3 = 3 \rightarrow x_1 = 3 - 7x_3 \)
- \( x_2 - 3x_3 = 2 \rightarrow x_2 = 2 - 3x_3 \)

Hence the general solutions for this system of equation is:

\[ X = (x_1, x_2, x_3) = (3 - 7x_3, 2 - 3x_3, x_3) = (3, 2, 0) + x_3(-7, -3, 1) \]
Problem 3.
We can find the set of independent vectors in set \( V \) (or basis of \( W \)) by applying \texttt{rref} operation on matrix 
\[ A = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix} \]  
where the vectors \( v_1, v_2, v_3, v_4 \) are the columns of the matrix \( A \).

Using \texttt{matlab} we have:

\[
\begin{align*}
\text{>> } A &= \begin{bmatrix} 3 & -1 & 6 & 2; 2 & 2 & 4 & 4; 1 & 2 & 2 & 3; 1 & 1 & 2 & 2 \end{bmatrix} \\
A & = \\
3 & -1 & 6 & 2 \\
2 & 2 & 4 & 4 \\
1 & 2 & 2 & 3 \\
1 & 1 & 2 & 2 \\
\text{>> } \text{rref}(A) \\
\text{ans} & = \\
1 & 0 & 2 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{align*}
\]

Which is obvious that the first and second columns can be considered as the set of base vectors for 
columns of matrix \texttt{rref}(A). Accordingly \{v_1, v_2\} is a basis for \( W \).
problem 4.

1. Using matlab we have:

   ```
   >> A=[1 1 1 1;1 -1 2 3; 1 1 4 9; 1 -1 8 27]
   A =
   1   1   1   1
   1  -1   2   3
   1   1   4   9
   1  -1   8  27
   >> rref(A)
   ans =
   1   0   0   0
   0   1   0   0
   0   0   1   0
   0   0   0   1
   ```

   Obviously the columns of matrix \(rref(A)\) are independent and accordingly the original columns of matrix \(A\) are independent.

2. \(b\) is the linear combination of vectors \(\{w_1, w_2, w_3, w_4\}\) if the system of equation \(Ax = b\) has some solution, where \(A\) is the matrix generated from \(\{w_1, w_2, w_3, w_4\}\) as columns and \(x^T = [x_1, x_2, x_3, x_4]\) is a vector in \(\mathbb{R}^4\). Setting up the augmented matrix \(AA = [A \ b]\) we have:

   ```
   >> A=[1 1 1 4; 1 -1 2 3; 1 1 4 9 2; 1 -1 8 27 1]
   A =
   1   1   1   4
   1  -1   2   3
   1   1   4   9
   1  -1   8  27
   >> rref(A)
   ans =
   1   0   0   0   5.0
   0   1   0   0   0.083
   0   0   1   0  -1.33
   0   0   0   1   0.25
   ```

   Which means: \(5.0w_1 + 0.083w_2 - 1.33w_3 + 0.25w_4 = b\)
problem 5.

1. Using Matlab/Octave:

>> S = [0 1 0 0 0; 0 0 1 0 0; 0 0 0 1 0; 0 0 0 0 1; 0 0 0 0 0]

S =

0 1 0 0 0
0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
0 0 0 0 0

To find $S^k$ for $S = 2, ..., 6$:

>> S^2
ans =

0 0 1 0 0
0 0 0 1 0
0 0 0 0 1
0 0 0 0 0
0 0 0 0 0

>> S^4
ans =

0 0 0 0 1
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0

>> S^6
ans =

0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
0 0 0 0 0
Problem 6.
In order to check if and square matrix $A_{n \times n}$ has an inverse or not, we set up the augmented matrix $AA = [A_{n \times n} \ | \ I_{n \times n}]$ and apply the rref operation on it. Matrix $A$ is invertible if and only if we end up with a matrix like $AA' = [I_{n \times n} \ | \ A'_{n \times n}]$ (namely the left box is the identity matrix), and matrix $A'$ is the inverse of $A$.

1. Setting up the augmented matrix $AA$ and applying rref using Matlab/Octave we have:

```matlab
>> AA = [1 0 -2 1 0 0; -3 1 4 0 1 0; 2 -3 4 0 0 1]
AA =
    1     0    -2     1     0     0
   -3     1     4     0     1     0
    2    -3     4     0     0     1
>> rref(AA)
ans =
     1.0000         0         0    8.0000    3.0000    1.0000
     0    1.0000         0   10.0000    4.0000    1.0000
     0         0    1.0000    3.5000    1.5000    0.5000
```

Accordingly the matrix $A$ is invertible and the inverse is:

$$
\begin{bmatrix}
8 & 0 & 0 \\
10 & 4 & 1 \\
7 & 3 & 1 \\
\end{bmatrix}
$$
2. Setting up the augmented matrix $AA$ and applying rref using Matlab/Octave we have:

$$AA = \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & 0 \\ -4 & -7 & 3 & 0 & 1 & 0 \\ -2 & -6 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$>> rref(AA)$$

$$\begin{bmatrix} 1.0000 & 0 & 1.0000 & 0 & -0.6000 & 0.7000 \\ 0 & 1.0000 & -1.0000 & 0 & 0.2000 & -0.4000 \\ 0 & 0 & 0 & 1.0000 & 0.2000 & 0.1000 \end{bmatrix}$$

As you see the left block after applying reduced echelon form will not result in identity matrix, which means the original matrix $A$ is not invertable.