Some Questions and Exercises About Matrices
Adding and Subtracting

**Question 1** Let $A$ be an $m \times n$ matrix and $B$ a $p \times q$ matrix. Then the addition $A + B$, and the subtraction $A - B$, are defined iff . . . ?

**Question 2** Let $A$, $B$ and $C$ be matrices of the appropriate dimensions.

1. Is matrix addition *commutative*, *i.e.*, $A + B = B + A$?

2. Is matrix addition *associative*, *i.e.*, $A + (B + C) = (A + B) + C$?

**Question 3** Repeat the preceding question for matrix subtraction.
Multiplication

**Question 4** Let $B$ be an $m \times p$ matrix and $A$ a $q \times n$ matrix. Then the product $BA$ is defined iff . . . ? What is the dimension of the resulting matrix?

**Question 5** Let $B$ be an $m \times p$ matrix, $A$ a $p \times n$ matrix, and $\vec{x}$ a vector with $n$ entries. Is it the case that $B(A\vec{x}) = (BA)\vec{x}$?

**Question 6** Let $C$ be a $k \times m$ matrix, $B$ an $m \times p$ matrix, and $A$ a $p \times n$ matrix. Is it the case that $C(BA) = (CB)A$ ?

**Question 7** Let $A$ be a $m \times n$ matrix and $D$ a $p \times q$ matrix. Let $B$ and $C$ be $n \times p$ matrices.

- Is it the case that $A(B + C) = AB + AC$ ?
- Is it the case that $(B + C)D = BD + CD$ ?
Multiplication

**Question 8** Let $B$ be a $k \times \ell$ matrix and $A$ a $\ell \times k$ matrix.

- What are the dimensions of the product $BA$?
- What are the dimensions of the product $AB$?
- Give examples of $B$ and $A$ such that $BA \neq AB$.

**Question 9** Let $A$ and $B$ be $2 \times 2$ matrices:

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}
\]

1. Write the $2 \times 2$ matrix resulting from multiplying $AB$.

2. Give nontrivial sufficient conditions in order that $AB = BA$. 
Different Kinds of Square Matrices

Definition 10 (Square Matrices) Let $A$ be a $n \times n$ square matrix:

$$A = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{bmatrix}$$

- $A$ is a symmetric matrix iff $a_{ij} = a_{ji}$ for all $1 \leq i, j \leq n$.
- $A$ has a constant diagonal iff $a_{11} = a_{22} = \cdots = a_{nn}$.
- $A$ is a diagonal matrix iff $a_{ij} = 0$ for all $i \neq j$.
- $A$ is an upper triangular matrix iff $a_{ij} = 0$ for all $i > j$.
- $A$ is an lower triangular matrix iff $a_{ij} = 0$ for all $i < j$. 
Multiplication

**Exercise 11** Consider arbitrary $2 \times 2$ symmetric matrices $A$ and $B$, each with constant diagonal.

1. Show that $AB = BA$.

2. Show that $A + B$ is a symmetric matrix with a constant diagonal.

3. Show that $A - B$ is a symmetric matrix with a constant diagonal.

4. Show that $AB$ is a symmetric matrix with a constant diagonal.

**Exercise 12** Show that only parts 1, 2, and 3 in Exercise 11 hold in general for $3 \times 3$ symmetric matrices with constant diagonals. State a sufficient condition so that part 4 also holds.

**Exercise 13** Repeat Exercise 11 for $n \times n$ upper triangular matrices, for an arbitrary $n \geq 2$. 
Theorem 14 Let $B$ be an $m \times p$ matrix and $A$ a $p \times n$ matrix. Let the columns of $A$ be $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ each with $p$ entries. Then the matrix product $BA$ is:

$$BA = B \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} = \begin{bmatrix} B\vec{v}_1 & B\vec{v}_2 & \cdots & B\vec{v}_n \end{bmatrix}$$

That is, to compute $BA$, we multiply $B$ with the $n$ column vectors of $A$ and combine the resulting $n$ column vectors.
Entries of the Matrix Product

**Theorem 15** Let $B$ be an $m \times p$ matrix and $A$ a $p \times n$ matrix. The $(i, j)$-th entry of the product $BA$ is the dot product of the $i$-th row of $B$ and the $j$-th column of $A$.

Specifically, if $B = (b_{ij})_{1 \leq i \leq m, 1 \leq j \leq p}$ and $A = (a_{k\ell})_{1 \leq k \leq p, 1 \leq \ell \leq n}$, then the $i$-th row of $B$ and the $j$-th column of $A$ are:

$$
\begin{bmatrix} b_{i1} & b_{i2} & \cdots & b_{ip} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{bmatrix}
$$

and their dot product is:

$$
b_{i1}a_{1j} + b_{i2}a_{2j} + \cdots + b_{ip}a_{pj} = \sum_{1 \leq k \leq p} b_{ik}a_{kj}
$$