Dot Product of Vectors

Let \( \vec{t} = (t_1, \ldots, t_n) \) and \( \vec{u} = (u_1, u_2, \ldots, u_n) \) be vectors of the same dimension \( n \geq 1 \) in \( \mathbb{R}^n \). The dot product of \( \vec{t} \) and \( \vec{u} \) is:

\[
\vec{t} \cdot \vec{u} = t_1 \cdot u_1 + t_2 \cdot u_2 + \cdots + t_n \cdot u_n
\]

If \( \vec{t} \) and \( \vec{u} \) are represented by \( n \times 1 \) column matrices, their dot product is:

\[
\vec{t} \cdot \vec{u} = (\vec{t})^T \vec{u} = \begin{bmatrix} t_1 & t_2 & \cdots & t_n \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \sum_{1 \leq i \leq n} t_i \cdot u_i
\]

**FACT:** \( \vec{t} \) and \( \vec{u} \) are perpendicular (or orthogonal) iff \( \vec{t} \cdot \vec{u} = 0 \).

Confirm the use of “perpendicular” (or “orthogonal”) by considering a few examples in \( \mathbb{R}^2 \) and \( \mathbb{R}^3 \).

What does it be mean for two vectors in \( \mathbb{R}^1 = \mathbb{R} \) to be “perpendicular”?
Dot Product of Vectors

**Example 1** The $n$ standard unit vectors

\[(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), (0, 0, 1, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)\]

are pairwise orthogonal.

**Example 2** Are the vectors $\vec{t}$ and $\vec{u}$ from the origin $O$ to the points $(4, 2)$ and $(-1, 2)$ in $\mathbb{R}^2$ orthogonal?

\[
\vec{t} \cdot \vec{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \end{bmatrix} = 4 \cdot (-1) + 2 \cdot 2 = 0
\]

**Example 3** Are the vectors $\vec{t}$ and $\vec{u}$ from the origin $O$ to the points $(1, -1, 2)$ and $(2, 4, 1)$ in $\mathbb{R}^3$ orthogonal?

\[
\vec{t} \cdot \vec{u} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}^T \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = 2 \cdot 1 + (-1) \cdot 4 + 2 \cdot 1 = 0
\]
Dot Product of Vectors

Example 4 Let $\vec{t} \in \mathbb{R}^2$ be the vector joining the origin $(0,0)$ to $(3,2)$. We want to write an equation for the line $L$ perpendicular to $\vec{t}$ at the origin. Consider an arbitrary point $(x,y)$ on $L$. The vector joining the origin $(0,0)$ to $(x,y)$ must satisfy:

$$\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3 \cdot x + 2 \cdot y = 0$$

The desired equation for $L$ is $3 \cdot x + 2 \cdot y = 0$.

Example 5 Let $\vec{t} \in \mathbb{R}^3$ be the vector joining the origin $(0,0,0)$ to $(1,4,8)$. We want to write an equation for the plane $P$ perpendicular to $\vec{t}$ at the origin. Consider an arbitrary point $(x,y,z)$ on $P$. The vector joining the origin $(0,0,0)$ to $(x,y,z)$ must satisfy:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 1 \\ 4 \\ 8 \end{bmatrix} = x + 4 \cdot y + 8 \cdot z = 0$$

The desired equation for $P$ is $x + 4 \cdot y + 8 \cdot z = 0$. 

4
Length (= Norm) of Vectors

The length (or norm) of a vector \( \vec{t} \in \mathbb{R}^n \) is \( \|\vec{t}\| = \sqrt{\vec{t} \cdot \vec{t}} \)

**Example 6** What is the length of the vector \( \vec{t} \) joining the origin \( \mathbf{O} \) to the point \((3, 4)\)?

\[
\|\vec{t}\| = \sqrt{(3, 4) \cdot (3, 4)} = \sqrt{25} = 5
\]

Draw the vector on the Cartesian plane and confirm the answer by geometric reasoning.

**Example 7** What is the length of the vector \( \vec{t} \) joining the origin \( \mathbf{O} \) to the point \((1, 4, 8)\)?

\[
\|\vec{t}\| = \sqrt{(1, 4, 8) \cdot (1, 4, 8)} = \sqrt{1 + 16 + 64} = \sqrt{81} = 9
\]

Can you draw the vector in the 3D Cartesian space and confirm the answer by geometric reasoning?
Linearly Independent Vectors

Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$, written as a sequence of $k$ vectors. The following definitions are lifted from different places in the book:

- $\vec{v}_1$ is redundant iff
  $$\vec{v}_1 = \mathbf{0} = (0,0,\ldots,0)$$

- $\vec{v}_j$ for $2 \leq j \leq k$ is redundant iff $\vec{v}_j$ is a linear combination of the vectors preceding it:
  $$\vec{v}_j = a_1 \vec{v}_1 + \cdots + a_{j-1} \vec{v}_{j-1}$$

- $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent iff none of them is redundant.

- $\vec{v}_1, \ldots, \vec{v}_k$ are linearly dependent iff at least one of them is redundant.
Linearly Independent Vectors

**Proposition 8** Let $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^n$. If $\vec{v}_1$ is nonzero, and if each of the vectors $\vec{v}_j$ for $j \geq 2$ has a nonzero entry in a position where all the preceding vectors $\vec{v}_1, \ldots, \vec{v}_{j-1}$ have a 0, then $\vec{v}_1, \ldots, \vec{v}_k$ are linearly independent.

**Proof:** Can you prove the proposition?

**Remark:** The proposition gives us a quick test for linear independence. This test depends on the order in which the $k$ vectors are listed. To apply it, it may be necessary to re-order the vectors.

**Example 9** The following vectors are linearly independent by Proposition 8:

- $\vec{v}_1 = (7, 0, 4, 0, 1, 9, 0)$ nonzero vector and therefore non-redundant
- $\vec{v}_2 = (6, 0, 7, 1, 4, 8, 0)$ non-redundant because of 4-th entry
- $\vec{v}_3 = (5, 0, 6, 2, 3, 1, 7)$ non-redundant because of 7-th entry
- $\vec{v}_4 = (4, 5, 3, 3, 2, 2, 4)$ non-redundant because of 2-nd entry

Note that we can **not** apply the test of Proposition 8 if $\vec{v}_3$ is listed after $\vec{v}_4$, *i.e.*, as $\vec{v}_1, \vec{v}_2, \vec{v}_4, \vec{v}_3$. 
Unit Vectors

$\vec{t} \in \mathbb{R}^n$ is a unit vector iff its length is 1, i.e., iff $\|\vec{t}\| = 1$.

Proposition 10 If $\vec{t} \in \mathbb{R}^n$ is nonzero, then $\vec{u} = (1/\|\vec{t}\|)\vec{t}$ is a unit vector.

Proof: $\vec{u} \cdot \vec{u} = ((1/\|\vec{t}\|)\vec{t}) \cdot ((1/\|\vec{t}\|)\vec{t}) = ((1/\|\vec{t}\|)^2 \vec{t} \cdot \vec{t}) = ((1/\vec{t} \cdot \vec{t}) \vec{t} \cdot \vec{t}) = 1$

Example 11 The $n$ standard unit vectors in $\mathbb{R}^n$:

$(1,0,0,\ldots,0), (0,1,0,\ldots,0), (0,0,1,\ldots,0), \ldots, (0,0,0,\ldots,1)$

are all of length 1, linearly independent, and mutually orthogonal.