Dimension of a Subspace of $\mathbb{R}^n$
Some Theorems

**Theorem 1 (A Subspace Always Has a Basis)**

Let $V$ be a subspace of $\mathbb{R}^n$. Let $m$ be the largest number of linearly independent vectors we can find in $V$.

(We always have $1 \leq m \leq n$. Why?)

Choose any such $m$ linearly independent vectors, say, $\vec{v}_1, \ldots, \vec{v}_m$.

Then $\text{span}(\vec{v}_1, \ldots, \vec{v}_m) = V$, which implies that $\vec{v}_1, \ldots, \vec{v}_m$ is a basis for $V$.

**Exercise 2**

Let $V$ be a subspace of $\mathbb{R}^n$. Show that $V$ is the image of some matrix.

**Theorem 3 (Number of Vectors in a Basis)**

All bases of a subspace $V$ of $\mathbb{R}^n$ have the same number of vectors.
Definition 4 (Dimension of a Subspace of $\mathbb{R}^n$)
Let $V$ be a subspace of $\mathbb{R}^n$. The number of vectors in a basis of $V$ is uniquely defined (by the two theorems on the preceding slide) and called the dimension of $V$, denoted $\dim(V)$.

Theorem 5 (Independent Vectors and Spanning Vectors in a Subspace)

Let $V$ be a subspace of $\mathbb{R}^n$ and $\dim(V) = m$. Then:

a. We can find at most $m$ linearly independent vectors in $V$.

b. We need at least $m$ vectors to span $V$.

c. $m$ vectors in $V$ are linearly independent iff they form a basis of $V$.

d. $m$ vectors in $V$ span $V$ iff they form a basis of $V$. 
**Theorem 6 (Using rref to construct a basis of the image)**

_to construct a basis of \( \text{im}(A) \), pick the column vectors of \( A \) that correspond to the column of rref(\( A \)) containing the leading 1's._

**Example 7**

Find the kernel (or, equivalently, the null space) of the linear transformation \( T \) represented by the following matrix \( A \):

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & 2 & 3 \\
1 & 2 & 3
\end{bmatrix}
\]

We have to find all vectors \( \vec{x} = (x_1, x_2, x_3) \) such that \( A \vec{x} = \vec{0} = (0, 0) \). It suffices to find the rref of the augmented matrix \([ A | \vec{0}]\):

\[
\text{rref}( [ A | \vec{0}] ) = \text{rref}( \begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 2 & 3 & 0 \\
0 & 1 & 2 & 0
\end{bmatrix} ) = \begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Translating back into linear equations, we get:

\[
x_1 - x_3 = 0 \quad \text{i.e.} \quad x_1 = x_3 \\
x_2 + x_3 = 0 \quad \text{i.e.} \quad x_2 = -2x_3
\]
Hence, taking \( x_3 = a \) where \( a \) is an arbitrary constant, every vector of the form \((a, -2a, a)\) is in the kernel of \( A \), i.e.:

\[
\ker(A) = \{ (a, -2a, a) \mid a \in \mathbb{R} \} 
\]

Stated differently, \( \ker(A) \) is the line spanned by the vector \((1, -2, 1)\) in \( \mathbb{R}^3 \).

\( \ker(A) \) is a vector subspace (not just a subset) of \( \mathbb{R}^3 \) and its dimension is 1, reflecting the fact there is only one “degree of freedom” represented by the single parameter \( a \) in the definition of \( \ker(A) \).
Example 8
Find the kernel (or, equivalently, the null space) of the linear transformation $T$ represented by the following matrix $A$:

$$A = \begin{bmatrix}
1 & 2 & 2 & -5 & 6 \\
-1 & -2 & -1 & 1 & -1 \\
4 & 8 & 5 & -8 & 9 \\
3 & 6 & 1 & 5 & -7
\end{bmatrix}$$

We proceed as in the preceding example: We find the rref of the augmented matrix $[A | \vec{0}]$:

$$\text{rref}([A | \vec{0}]) = \begin{bmatrix}
1 & 2 & 0 & 3 & -4 & 0 \\
0 & 0 & 1 & -4 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

Translating back into linear equations, we get:

$$x_1 + 2x_2 + 3x_4 - 4x_5 = 0 \quad \text{i.e.} \quad x_1 = -2x_2 - 3x_4 + 4x_5$$
$$x_3 - 4x_4 + 5x_5 = 0 \quad \text{i.e.} \quad x_3 = 4x_4 - 5x_5$$

Hence, taking $x_2 = a$, $x_4 = b$, and $x_5 = c$, where $a$, $b$, and $c$, are arbitrary
constants, every vector of the form
\[ (-2a - 3b + 4c, a, 4b - 5c, b, c) \]
is in the kernel of \( A \), i.e.:
\[ \ker(A) = \{ (-2a - 3b + 4c, a, 4b - 5c, b, c) \mid a, b, c \in \mathbb{R} \} \]
which can also be written as:
\[ \ker(A) = \{ a(-2,1,0,0,0) + b(-3,0,4,1,0) + c(4,0,-5,0,1) \mid a, b, c \in \mathbb{R} \} \]

Note that \( \operatorname{im}(A) = \text{Col}(A) \) has dimension 2 as a subspace of \( \mathbb{R}^4 \), because only two of its columns are linearly independent, while \( \operatorname{Nul}(A) = \ker(A) \) has dimension 3 (i.e., 3 “degrees of freedom”) as a subspace of \( \mathbb{R}^5 \).
Theorem 9 (Dimension of the Image)
For any \( m \times n \) matrix \( A \), we have \( \text{dim}(\text{im}(A)) = \text{rank}(A) \),
or equivalently \( \text{dim}(\text{Col}(A)) = \text{rank}(A) \).

Theorem 10 ("Rank-Nullity" Theorem)
For any \( m \times n \) matrix \( A \), we have \( \text{dim}(\text{ker}(A)) + \text{dim}(\text{im}(A)) = n \),
or equivalently \( \text{dim}(\text{Nul}(A)) + \text{dim}(\text{Col}(A)) = n \).

Theorem 11 (Rank Theorem – page 156 and page 233 in [LAA])
For any \( m \times n \) matrix \( A \), we have \( \text{dim}(\text{Nul}(A)) + \text{rank}(A) = n \).

Example 12
Check the equalities stated in Theorems 9, 11, and 10, with the matrices in Examples 7 and 8.
**Theorem 13 (Invertible Square Matrices)**

Let $A$ be a $n \times n$ matrix. The following are equivalent:

1. $A$ is invertible.
2. The linear system $A\overrightarrow{x} = \overrightarrow{b}$ has a unique solution for $\overrightarrow{x}$, for every $\overrightarrow{b} \in \mathbb{R}^n$.
3. $\text{rref}(A) = I_n$, the identity $n \times n$ matrix.
4. $\text{rank}(A) = n$.
5. $\text{im}(A) = \text{Col}(A) = \mathbb{R}^n$.
6. $\text{ker}(A) = \text{Nul}(A) = \{\overrightarrow{0}\}$.
7. The column vectors of $A$ form a basis of $\mathbb{R}^n$.
8. The column vectors of $A$ span $\mathbb{R}^n$.
9. The column vectors of $A$ are linearly independent.