Equivalent Conditions of
Invertibility and Non-Invertibility
The following are equivalent statements* for $n \times n$ matrix $A$

1. $A$ is invertible, i.e., $A^{-1}$ exists.

2. $A$ is nonsingular.

3. The linear system $A\vec{x} = \vec{0}$ has exactly one solution $\vec{x} = \vec{0}$.

4. The linear system $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = A^{-1}\vec{b}$.

5. $\text{rref}(A) = I_n$, i.e., $A$ has $n$ (nonzero) pivots.

6. $\text{rank}(A) = n$.

7. $\text{im}(A) = \text{Col}(A) = \mathbb{R}^n$.

8. $\text{im}(A^T) = \text{Col}(A^T) = \mathbb{R}^n$.

*Good review for end-of-term exam: Read carefully each statement and give a reason why it is equivalent to invertibility.
9. \( \ker(A) = \text{Nul}(A) = \{ \vec{0} \} \)

10. \( \ker(A^T) = \text{Nul}(A^T) = \{ \vec{0} \} \).

11. The linear transformation represented by \( A \) is one-one (i.e., injective).

12. The linear transformation represented by \( A \) is onto (i.e., surjective).

13. The column vectors of \( A \) form a basis for \( \mathbb{R}^n \).

14. The row vectors of \( A \) form a basis for \( \mathbb{R}^n \).

15. The column vectors of \( A \) span \( \mathbb{R}^n \).

16. The row vectors of \( A \) span \( \mathbb{R}^n \).

17. The column vectors of \( A \) are linearly independent.

18. The row vectors of \( A \) are linearly independent.

19. \( \det A \neq 0 \).

20. All eigenvalues of \( A \) are nonzero (not yet covered in lecture).
The following are equivalent statements* for $n \times n$ matrix $A$

1. $A$ is not invertible, i.e., $A^{-1}$ is not defined.

2. $A$ is singular.

3. The linear system $Ax = \vec{0}$ has infinitely many solutions.

4. The linear system $Ax = \vec{b}$ has no solution or infinitely many.

5. $\text{rref}(A)$ has at least one zero row.

6. $\text{rank}(A) < n$.

7. $\text{im}(A) = \text{Col}(A) \subset \mathbb{R}^n$.

8. $\text{im}(A^T) = \text{Col}(A^T) \subset \mathbb{R}^n$.

*Good review for end-of-term exam: Read carefully each statement and give a reason why it is equivalent to non-invertibility.
9. \( \ker(A) = \text{Nul}(A) \supseteq \{\mathbf{0}\} \).

10. \( \ker(A^T) = \text{Nul}(A^T) \supseteq \{\mathbf{0}\} \).

11. The linear transformation represented by \( A \) is not one-one (i.e., injective).

12. The linear transformation represented by \( A \) is not onto (i.e., surjective).

13. The column vectors of \( A \) do not form a basis for \( \mathbb{R}^n \).

14. The row vectors of \( A \) do not form a basis for \( \mathbb{R}^n \).

15. The column vectors of \( A \) do not span \( \mathbb{R}^n \).

16. The row vectors of \( A \) do not span \( \mathbb{R}^n \).

17. The column vectors of \( A \) dependent.

18. The row vectors of \( A \) dependent.

19. \( \det A = 0 \).

20. At least one eigenvalue of \( A \) is 0 (not yet covered in lecture).
Other specialized topics from first half of the semester

1. Correspondence between geometric transformations in 2D (and 3D) and $2 \times 2$ (and $3 \times 3$) matrices.

2. Correspondence between elementary row operations and elementary matrices (including permutation matrices).

3. Cramer’s rule.


5. Matrix factorization and LU decomposition.