How to Find the Matrix that Represents a Linear Transformation

Reminder:

A function \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is called a \textit{linear transformation} iff two conditions:

1. \( T \) preserves vector addition, \textit{i.e.}, \( T(u + v) = T(u) + T(v) \) for all vectors \( u, v \in \mathbb{R}^n \).
2. \( T \) preserves scalar multiplication, \textit{i.e.}, \( T(c \cdot u) = c \cdot T(u) \) for all vector \( u \in \mathbb{R}^n \) and scalar \( c \in \mathbb{R} \).

\textbf{Notation}: Throughout this handout, “·” denotes scalar multiplication, not the \textit{dot product} (called \textit{inner product} in \cite{Lay}) of vectors.

\textbf{Many Examples:}

The rest of this handout consists of many examples, some easy and some more difficult. We often omit the obvious details, especially in the easy examples. Two examples are about \textit{non-linear} transformations (Examples 5 and 6), to build up intuition and contrast them with \textit{linear} transformations.

\textbf{EXAMPLE 1} : The transformation \( T_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( T_1(a, b) = (b, a) \) is linear. Check the two conditions defining a linear transformation yourself! By inspection, because it is a very simple transformation, its matrix representation is:

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

\textbf{EXAMPLE 2} : The transformation \( T_2 : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) such that \( T_2(a, b) = (b, a, a + b) \) is linear. Check the two conditions defining a linear transformation yourself! By inspection, inspired by Example 1, its matrix representation is:

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}
\]

\textbf{EXAMPLE 3} : The transformation \( T_3 : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) such that \( T_3(a, b, c) = (a + c, b + c) \) is linear. Check the two conditions defining a linear transformation yourself! By inspection, inspired by Examples 1 and 2, its matrix representation is:

\[
A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}
\]

\textbf{EXAMPLE 4} : The transformation \( T_4 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( T_4(a, b) = (2 \cdot b, 2 \cdot a) \) is “almost” like transformation \( T_1 \) in Example 1. In fact, we can write:

\[
T_4(a, b) = 2 \cdot T_1(a, b).
\]

Is \( T_4 \) linear? Yes, it is. Check the two conditions defining a linear transformation yourself! By inspection, inspired by the preceding examples, its matrix representation is:

\[
A_4 = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}
\]
EXAMPLE 5: The transformation \( T_5 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) such that \( T_5(a, b) = (1 + b, 1 + a) \) is also “almost” like transformation \( T_1 \) in Example 1. In fact, we can write:

\[
T_5(a, b) = (1, 1) + T_1(a, b).
\]

Is \( T_5 \) linear? NO, it is not. See Example 6 for hints how to show this.\(^1\)

EXAMPLE 6: The transformation \( T_6 : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) such that \( T_6(a, b) = (b, a, 1 + a + b) \) is a variation of \( T_2 \) in Example 2. It is NOT linear. We can decompose \( T_6 \) in a way similar to the decomposition of \( T_5 \) in Example 5:

\[
T_6(a, b) = (0, 0, 1) + T_2(a, b).
\]

To show that \( T_6 \) is not a linear transformation, consider the vector \( \mathbf{u} = (5, 10) \in \mathbb{R}^2 \) and the scalar \( c = 2 \in \mathbb{R} \). Then:

\[
T_6(c \cdot \mathbf{u}) = T_6(2 \cdot (5, 10)) = T_6(10, 20) = (20, 10, 31),
\]

whereas:

\[
c \cdot T_6(\mathbf{u}) = 2 \cdot T_6(5, 10) = 2 \cdot (10, 5, 16) = (20, 10, 32).
\]

Clearly, \( (20, 10, 31) \neq (20, 10, 32) \), so that \( T_6(c \cdot \mathbf{u}) \neq c \cdot T_6(\mathbf{u}) \), i.e., \( T_6 \) does not preserve scalar multiplication.

EXAMPLE 7: Let \( T_7 : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be the transformation that rotates the 2D \((x, y)\)-plane by the angle \( \theta \) in the clockwise direction. From trigonometry, the new coordinates \( x' \) and \( y' \) can be expressed in terms of the original coordinates \( x \) and \( y \) according to the formulas:

\[
x' = x \cos(\theta) + y \sin(\theta) \quad \text{and} \quad y' = -x \sin(\theta) + y \cos(\theta).
\]

The transformation \( T_7 \) is linear, and the matrix representing it is:

\[
A_7 = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

As an aside, illustrating the convenience of matrix algebra, consider the matrix product \( A_7 A_7 \), which defines two consecutive clockwise rotations by the angle \( \theta \). Based on geometric principles of plane geometry, it must be the case that:

\[
A_7 A_7 = \begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
-\sin(2\theta) & \cos(2\theta)
\end{bmatrix}
\]

Carrying out the matrix product \( A_7 A_7 \), we must also have:

\[
A_7 A_7 = \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix} \begin{bmatrix}
\cos(\theta) & \sin(\theta) \\
-\sin(\theta) & \cos(\theta)
\end{bmatrix} = \begin{bmatrix}
\cos^2(\theta) - \sin^2(\theta) & 2 \cos(\theta) \sin(\theta) \\
-2 \cos(\theta) \sin(\theta) & \cos^2(\theta) - \sin^2(\theta)
\end{bmatrix}
\]

Hence, comparing the product \( A_7 A_7 \) according to its geometric meaning in the plane and the product \( A_7 A_7 \) according to matrix algebra, we obtain that:

\[
\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) \quad \text{and} \quad \sin(2\theta) = \cos(\theta) \sin(\theta)
\]

We have thus obtained the so-called double-angle identities of trigonometry using matrix algebra!

\(^1\)Note the shape of the transformation \( T_5 \), which is an example of a transformation \( U : \mathbb{R}^n \rightarrow \mathbb{R}^m \) such that \( U(v) = c + T(v) \) where \( v \) ranges over \( \mathbb{R}^n \) and \( c \) is a particular constant vector in \( \mathbb{R}^m \). A transformation such as \( U \) is called an affine transformation. Every linear transformation is an affine transformation, but not vice-versa. The transformation \( T_6 \) in Example 6 is another affine transformation which is not linear.
Hence, by Fact 2, the matrix representing vectors $v_1, v_2, \ldots, v_n$ be a vector in $\mathbb{R}^n$. We then have the following vector equation:

$$A v = v_1 \cdot u_1 + v_2 \cdot u_2 + \cdots + v_n \cdot u_n.$$  

**Proof:** This is easy, left to you to do. It is a nice review of basic matrix operations.

For the next fact, recall what it means for a set of vectors $B = \{e_1, e_2, \ldots, e_n\}$ to be the standard basis of $\mathbb{R}^n$:

$$e_1 = (1, 0, 0, \ldots, 0, 0), \quad e_2 = (0, 1, 0, \ldots, 0, 0), \quad \ldots, \quad e_n = (0, 0, 0, \ldots, 1).$$

**Fact 2** Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation defined in some precise way other than by a $m \times n$ matrix. Let $B = \{e_1, e_2, \ldots, e_n\}$ be the standard basis for the vector space $\mathbb{R}^n$.

If it is possible to determine what $T$ does to each member of $B$, i.e., we are somehow able to compute the vectors $T(e_1), T(e_2), \ldots, T(e_n)$ which are all in the space $\mathbb{R}^m$, then the $m \times n$ matrix representation of $T$ is:

$$A = \begin{bmatrix} T(e_1) | T(e_2) | \cdots | T(e_n) \end{bmatrix},$$

and the action of $T$ on an arbitrary vector $v = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ is:

$$T(v) = A v = \begin{bmatrix} T(e_1) | T(e_2) | \cdots | T(e_n) \end{bmatrix} v = v_1 \cdot T(e_1) + v_2 \cdot T(e_2) + \cdots + v_n \cdot T(e_n).$$

**Proof:** Based on Fact 1 above, this is also easy. Left to you.

**EXAMPLE 8** Let $T_8 : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation such that $T_8(a, b) = (a - b, a + b, a)$. Equivalently, we can write:

$$T_8 \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \\ a \end{bmatrix}.$$  

We can write the matrix $A_8$ representing $T_8$ by inspection, because $T_8$ is very simply defined here. But we choose to be lazy and mechanically use Fact 2, which requires computing $T_8(e_1)$ and $T_8(e_2)$:

$$T_8(e_1) = T_8 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad T_8(e_2) = T_8 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$  

Hence, by Fact 2, we have:

$$A_8 = \begin{bmatrix} T_8(e_1) & T_8(e_2) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}.$$  

**EXAMPLE 9** Let $T_9 : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that perpendicularly projects the standard unit vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$ onto the line defined by the equation $x = y$ in the 2-dimensional $(x, y)$-plane. In this case, plane geometry tells us that:

$$T_9(e_1) = T_9 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} \quad \text{and} \quad T_9(e_2) = T_9 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}.$$  

Make sure you understand the preceding assertion! Hence, by Fact 2, the matrix representing $T_9$ is:

$$A_9 = \begin{bmatrix} T_9(e_1) & T_9(e_2) \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$
EXAMPLE 10 Suppose \( T_{10} : \mathbb{R}^3 \to \mathbb{R}^3 \) is a linear transformation and we know its action on three linearly independent vectors in \( \mathbb{R}^n \), namely, \( u_1 = (1, 2, 3), u_2 = (2, 3, 4), \) and \( u_3 = (0, 2, 3) \):

\[
T_{10}(u_1) = T_{10} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T_{10}(u_2) = T_{10} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad T_{10}(u_3) = T_{10} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.
\]

The standard basis for \( \mathbb{R}^3 \) is:

\[
B = \{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}.
\]

By Fact 2 above, to compute the matrix \( A_{10} \) representing \( T_{10} \) it suffices to compute \( T_{10}(e_1), T_{10}(e_2), \) and \( T_{10}(e_3) \), and then put them together as the column vectors of the desired \( A_{10} \).

In order to proceed further and invoke Fact 2, the trick is to express each of \( e_1, e_2, \) and \( e_3 \), in terms of \( \{u_1, u_2, u_3\} \). Let’s first define \( e_1 \) in terms of \( \{u_1, u_2, u_3\} \), i.e., we want to express \( e_1 \) as a linear combination of \( \{u_1, u_2, u_3\} \). This means we want scalars \( c_1, c_2, \) and \( c_3 \), such that:

\[
e_1 = c_1 \cdot u_1 + c_2 \cdot u_2 + c_3 \cdot u_3.
\]

Inserted the numbers, the preceding becomes the following matrix equation:

\[
\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \cdot \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + c_3 \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}
\]

We next solve for \( \{c_1, c_2, c_3\} \) by breaking up the preceding matrix equation into a system of three linear equations:

\[
\begin{align*}
c_1 + c_2 &= 1 \\
2c_1 + 3c_2 + 2c_3 &= 0 \\
3c_1 + 4c_2 + 3c_3 &= 0
\end{align*}
\]

Using methods we have used all semester long (details omitted), the solution is \( c_1 = 1, c_2 = 0, \) and \( c_3 = -1 \).

Applying \( T_{10} \) to both sides of the matrix equation above, we obtain – notice how we can “push” \( T_{10} \) past the scalars \( c_1 = 1, c_2 = 0, \) and \( c_3 = -1 \), because we are given that \( T_{10} \) is a linear transformation:

\[
T_{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1 \cdot T_{10} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \cdot T_{10} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + c_3 \cdot T_{10} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = T_{10} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - T_{10} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}
\]

Proceeding in a similar way (all details left to you), we also find:

\[
T_{10} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix} \quad \text{and} \quad T_{10} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}
\]

Finally, the matrix \( A_{10} \) representing \( T_{10} \) is:

\[
A_{10} = \begin{bmatrix} 0 & -4 & 3 \\ -1 & 1 & 0 \\ -1 & 4 & -2 \end{bmatrix}
\]

EXAMPLE 11 Suppose \( T_{11} : \mathbb{R}^2 \to \mathbb{R}^2 \) is the transformation such that \( T_{11}(x) = \text{ref}_L(x) \) for every \( x \in \mathbb{R}^2 \), where \( L \) is a line through the origin \((0, 0)\) and \( \text{ref}_L(x) \) is the reflection of \( x \) about \( L \). (See Exercise 34, page 345, in [Lay], which shows that every reflection about a line is a linear transformation.)

We handle this example abstractly, i.e., without specifying the direction of the line \( L \). So, it may be a little harder to follow than the preceding examples, but the resulting analysis is more general, i.e., it applies to any reflection about any line through the origin \((0, 0)\).
We want to determine the matrix $A_{11}$ representing $T_{11}$. Let $\mathbf{u}$ be the unit vector along (i.e., co-linear with) line $L$. In 2D, $\mathbf{u} = (u_1, u_2)$ for some $u_1, u_2 \in \mathbb{R}$ such that $u_1^2 + u_2^2 = 1$. We have the following equalities:

\[
\ref_L(x) = 2(x \cdot \mathbf{u}) \mathbf{u} - x
\]

\[
= 2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)
= 2\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)
= 2(x_1u_1 + x_2u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
= \begin{bmatrix} (2u_1^2 - 1)x_1 + 2u_1u_2x_2 \\ 2u_1u_2x_1 + (2u_2^2 - 1)x_2 \end{bmatrix}
= \begin{bmatrix} (2u_1^2 - 1) & 2u_1u_2 \\ 2u_1u_2 & (2u_2^2 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\]

The desired matrix $A_{11}$ is therefore:

\[
(\dagger) \quad A_{11} = \begin{bmatrix} (2u_1^2 - 1) & 2u_1u_2 \\ 2u_1u_2 & (2u_2^2 - 1) \end{bmatrix}
\]

Because $u_1^2 + u_2^2 = 1$, we can simplify a little. If we add the entries along the main diagonal of $A$, we obtain:

\[
(2u_1^2 - 1) + (2u_2^2 - 1) = 2(u_1^2 + u_2^2) - 2 = 2 \cdot 1 - 2 = 0
\]

Thus, if we write $(2u_1^2 - 1) = a$, then $(2u_2^2 - 1) = -a$. Make sure you understand the two preceding assertions! Hence, also:

\[
u_1 = \sqrt{(1 + a)/2} \quad \text{and} \quad u_2 = \sqrt{(1 - a)/2}
\]

From the preceding, we get:

\[
2 \cdot u_1 \cdot u_2 = \sqrt{(1 + a) \cdot (1 - a)} = \sqrt{1 - a^2}
\]

Hence, if we write $2 \cdot u_1 \cdot u_2 = b$, then $b = (1 - a^2)^{1/2}$. This implies $a^2 + b^2 = 1$. Hence, the desired matrix $A_{11}$ is:

\[
(\dagger) \quad A_{11} = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}
\]

where $a^2 + b^2 = 1$. Note the form of $A_{11}$: It is an orthogonal matrix.

The converse of the preceding is also true, namely: If a matrix has the form of $A_{11}$, then it is a matrix representing a reflection about some line $L$.

**EXAMPLE 12** Repeat Example 11 when the line $L$ and the vector $x$ are in 3D. Call $A_{12}$ the matrix to be determined. We omit the details, which are very similar to those in Example 11 – except that it is more difficult to put $A_{12}$ in the simple form shown in Equation (\dagger). In 3D, you should be able to reach the following equalities:

\[
\ref_L(x) = 2(x \cdot \mathbf{u}) \mathbf{u} - x = \begin{bmatrix} (2u_1^2 - 1) & 2u_1u_2 & 2u_1u_3 \\ 2u_1u_2 & (2u_2^2 - 1) & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & (2u_3^2 - 1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\]

Hence, the desired matrix $A_{12}$ is:

\[
(\#) \quad A_{12} = \begin{bmatrix} (2u_1^2 - 1) & 2u_1u_2 & 2u_1u_3 \\ 2u_1u_2 & (2u_2^2 - 1) & 2u_2u_3 \\ 2u_1u_3 & 2u_2u_3 & (2u_3^2 - 1) \end{bmatrix}
\]

Equation (\#) in 3D is the counterpart of Equation (\dagger) in 2D. As pointed out already, it is a little more challenging to find the 3D counterpart of Equation (\dagger) in 2D.