Orthogonal Matrices,
Symmetric Matrices,
Semi-Definite Matrices, etc.
Equivalent statements about a \( n \times n \) matrix \( P \)

1. \( P \) is an orthogonal matrix.

2. \( P^T P = I_n \).

3. \( PP^T = I_n \).

4. The column vectors of \( P \) are an orthonormal basis of \( \mathbb{R}^n \).

5. The row vectors of \( P \) are an orthonormal basis of \( \mathbb{R}^n \).

6. \( P^{-1} = P^T \).

Given points 4 and 5 above, perhaps a better name for orthogonal matrices is orthonormal.

But we have to stick to the accepted terminology …
Examples of orthogonal \( n \times n \) matrices

Example 1

\[
P_1 = (1/7) \begin{bmatrix} 2 & 6 & 3 \\ 3 & 2 & -6 \\ 6 & -3 & 2 \end{bmatrix} \quad \text{(check it is orthogonal!)}
\]

Example 2

\[
P_2 = (1/2) \begin{bmatrix} 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \quad \text{(check it is orthogonal!)}
\]

Warning: A matrix with orthogonal columns (or orthogonal rows) need not be an orthogonal matrix. For example,

\[
\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}
\]

is not an orthogonal matrix, even though its column vectors are orthogonal and so are its row columns.
Equivalent ways of saying $n \times n$ matrix $A = (a_{i,j})$ is symmetric

1. “If we rotate $A$ about its main diagonal, the resulting matrix is $A$ again.”

2. $A = A^T$, i.e., $A$ is the same as its transpose.

3. For all $1 \leq i, j \leq n$, $a_{i,j} = a_{j,i}$. 
Examples of symmetric $n \times n$ matrices

Example 3

\[ A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix} \quad \text{rref}(A_1) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

The eigenvalues of $A_1$ with their algebraic multiplicities: 0, 0, 14.

Example 4

\[ A_2 = \begin{bmatrix} 1 & -3 & 3 \\ -3 & 1 & -1 \\ 3 & -1 & -1 \end{bmatrix} \quad \text{rref}(A_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]

The eigenvalues of $A_2$: $-2889/646$, $-1$, $2889/646$

Example 5

\[ A_3 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \end{bmatrix} \quad \text{rref}(A_3) = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

The eigenvalues of $A_2$ with their algebraic multiplicities: $-769/660$, 0, 0, $11329/660$. 

5
Properties of Symmetric Matrices

An $n \times n$ symmetric matrix $A$ has all of the following properties:

1. Counting eigenvalues with their algebraic multiplicities, $A$ always has $n$ eigenvalues, and they are all real.

2. We can always choose $n$ orthonormal eigenvectors of $A$.

3. $A$ is orthogonally diagonalizable: There is an orthogonal matrix $P$ s.t. $P^{-1}AP = D$ with $D$ a diagonal matrix with the eigenvalues of $A$ (with their multiplicities) on the diagonal.

4. Two eigenvectors of $A$ corresponding to two distinct eigenvalues are always orthogonal.

5. For every eigenvalue $\lambda$ of $A$, the algebraic multiplicity and the geometric multiplicity of $A$ are always equal.

Exercise 6 Check each of the 5 properties above for the symmetric matrices in Examples 3, 4, and 5.
More Properties of Symmetric Matrices

1. Symmetric matrices are closed under:
   - matrix addition,
   - scalar multiplication
     (but not matrix multiplication – counter-example, please?),
   - raising to integer power,
   - If non-singular, inversion.

Exercise 7

Let $A$ and $B$ be $n \times n$ symmetric matrices. Show that $A \ast B + B \ast A$ is symmetric matrix.
Special Cases of Symmetric Matrices

1. A symmetric matrix whose eigenvalues are all positive is called \textit{positive definite}.

2. A symmetric matrix whose eigenvalues are all negative is called \textit{negative definite}.

3. A symmetric matrix whose eigenvalues include both positive and negative numbers is called \textit{indefinite}.

Exercise 8

1. Suppose the \( n \) eigenvalues of a \( n \times n \) matrix are all strictly positive. Is \( A \) symmetric?
   \textit{Hint:} No. Find a \( 2 \times 2 \) counterexample.

2. Suppose the \( n \) eigenvalues of a \( n \times n \) matrix are all strictly negative. Is \( A \) symmetric?
   \textit{Hint:} No. Find a \( 2 \times 2 \) counterexample.
Classification of matrices with only real eigenvalues*

From the $n \times n$ matrix $A$, construct the $n \times n$ diagonal matrix $D$ by using the eigenvalues of $A$ with their algebraic multiplicities:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 \\ 0 & d_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & d_n \end{bmatrix}$$

$A$ (and $D$ constructed from $A$’s eigenvalues) is said to be:

1. **positive definite** iff $d_i > 0$ for every $1 \leq i \leq n$.

2. **negative definite** iff $d_i < 0$ for every $1 \leq i \leq n$.

3. **positive semidefinite** iff $d_i \geq 0$ for every $1 \leq i \leq n$.

4. **negative semidefinite** iff $d_i \leq 0$ for every $1 \leq i \leq n$.

5. **indefinite** iff $d_i > 0$ for some indices $i$, $1 \leq i \leq n$, and $d_i < 0$ for other indices.

*Many books, including [Lay], restrict this classification to symmetric matrices.
Examples of positive define and positive semidefinite

**Example 9** $A_4$ is positive definite:

$$A_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \quad \text{rref}(A_4) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of $A_4$ are: 2, $(2 - \sqrt{2})$, $(2 + \sqrt{2})$.

**Example 10** For what values of $a$, is $A_5$ positive definite?

$$A_5 = \begin{bmatrix} 2 & -1 & a \\ -1 & 2 & -1 \\ a & -1 & 2 \end{bmatrix} \quad \text{i.e., find } a \text{ such that rref}(A_5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigenvalues of $A_5$ are???

Are there values of $a > 2$ for which $A_5$ is positive definite???

**Example 11** $A_6$ is positive semidefinite:

$$A_6 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \quad \text{rref}(A_6) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

The eigenvalues of $A_6$ with their algebraic multiplicities: 3, 3, 0.