Solutions

**Question 1:** Let $A$ and $B$ be $2 \times 2$ symmetric matrices. Is it always the case that $A \ast B$ is a symmetric matrix? If **YES**, justify carefully in at most 2-3 lines. If **NO**, give a counter-example.

**Answer:** **NO.** Take $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 3 & 3 \end{bmatrix}$. Then $A \ast B = \begin{bmatrix} 8 & 9 \\ 10 & 12 \end{bmatrix}$, which is not symmetric.

**Question 2:** Let $A$ and $B$ be $2 \times 2$ symmetric matrices. Is it always the case that $A \ast B + B \ast A$ is a symmetric matrix? If **YES**, justify carefully in at most 2-3 lines. If **NO**, give a counter-example.

**Answer:** **YES.** Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and $B = \begin{bmatrix} x & y \\ y & z \end{bmatrix}$. Then

$$A \ast B = \begin{bmatrix} ax + by & ay + bz \\ bx + cy & by + cz \end{bmatrix} \quad \text{and} \quad B \ast A = \begin{bmatrix} ax + by & bx + cy \\ ay + bz & by + cz \end{bmatrix}$$

which implies the following symmetric matrix:

$$A \ast B + B \ast A = \begin{bmatrix} 2(ax + by) & (ay + bz) + (bx + cy) \\ (ay + bz) + (bx + cy) & 2(by + cz) \end{bmatrix}$$

Alternative answer: For any square matrix $X$, whether symmetric or not, the matrix $(X + X^T)$ is symmetric. Consider now $(A \ast B)^T = B^T \ast A^T$. Because $A$ and $B$ are symmetric, $A^T = A$ and $B^T = B$. Hence, $(A \ast B)^T = B \ast A$, so that also $(A \ast B) + (B \ast A) = (A \ast B) + (A \ast B)^T$, which is symmetric.

**Question 3:** For a $n \times n$ matrix $A$, is it always the case that if $A^2$ is invertible, then $A$ itself is also invertible? If **YES**, justify carefully in at most 2-3 lines. If **NO**, give a counter-example.

**Answer:** **YES.** If $A^2$ is invertible, there is a matrix $B$ such that $A^2B = I_n$. Because matrix multiplication is associative, $A(AB) = I_n$. This implies $AB$ is the inverse of $A$, which also implies that $A$ is invertible.

Alternative answer: $A^2$ invertible implies $\det(A^2) \neq 0$. But $\det(A^2) = \det(A) \det(A)$ (see Theorem 6, page 173, in [Lay]). If $\det(A) \det(A) \neq 0$, then $\det(A) \neq 0$, which implies that $A$ is invertible.