CS 511, Fall 2018, Handout 01

Propositional Logic:

From Truth Tables To Conjunctive Normal Forms (CNF) and Disjunctive Normal Forms (DNF)

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some familiar truth-tables:

logical “or” ($\vee$) and logical “and” ($\wedge$)

x	y	$x \vee y$
T	**T**	**T**
T	**F**	**T**
F	**T**	**T**
F	**F**	**F**

x	y	$x \wedge y$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**F**

logical “implication” ($\rightarrow$)

x	y	$x \rightarrow y$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

and similarly for “negation” ($\neg$) and many other logical connectives . . . .

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y
T	**T**
T	**F**
F	**T**
F	**F**

from *propositional formulas* to *truth-tables*

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x	y	$\neg x$
T	**T**	**F**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

from *propositional formulas* to *truth-tables*

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x	y	$\neg x$	$\neg y$
T	**T**	**F**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**F**	**T**	**T**

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$
T	**T**	**F**	**F**	**F**
T	**F**	**F**	**T**	**T**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**T**	**T**

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
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x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$
T	**T**	**F**	**F**	**F**	**T**
T	**F**	**F**	**T**	**T**	**F**
F	**T**	**T**	**F**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**

from *propositional formulas* to *truth-tables*

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- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional wff φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional wff φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.

from *propositional formulas* to *truth-tables*

consider propositional wff (**w**ell-**f**ormed **f**ormula): $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$:

- start with all the propositional atoms in the wff φ
- incrementally, consider each sub-wff of φ, from innermost to outermost

x	y	$\neg x$	$\neg y$	$x \to \neg y$	$y \vee \neg x$	$(x \to \neg y) \to (y \vee \neg x)$
T	**T**	**F**	**F**	**F**	**T**	**T**
T	**F**	**F**	**T**	**T**	**F**	**F**
F	**T**	**T**	**F**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**	**T**	**T**

- propositional wff φ is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes φ true.
- propositional wff φ is a **tautology** if **every** assignment of truth-values to the propositional atoms makes φ true.
- $\varphi \triangleq (x \to \neg y) \to (y \vee \neg x)$ is satisfiable, but is not a tautology.

example of a truth-table

x	y	z	$f(x,y,z)$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

input variables: $\{x,y,z\}$

output: $f(x,y,z)$

$f : \{\mathbf{T},\mathbf{F}\}\times\{\mathbf{T},\mathbf{F}\}\times\{\mathbf{T},\mathbf{F}\} \to \{\mathbf{T},\mathbf{F}\}$

or also $\quad f : \{\mathbf{T},\mathbf{F}\}^3 \to \{\mathbf{T},\mathbf{F}\}$

example of a truth-table

x	y	z	$f(x,y,z)$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

- f is a function which returns **T** iff exactly two of its three variables are assigned the value **T**
- there are many different ways of writing a formula for f

input variables: $\{x, y, z\}$

output: $f(x, y, z)$

$f : \{\mathbf{T},\mathbf{F}\}\times\{\mathbf{T},\mathbf{F}\}\times\{\mathbf{T},\mathbf{F}\} \to \{\mathbf{T},\mathbf{F}\}$

or also $f : \{\mathbf{T},\mathbf{F}\}^3 \to \{\mathbf{T},\mathbf{F}\}$

example of a truth-table

x	y	z	$f(x,y,z)$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

input variables: $\{x,y,z\}$

output: $f(x,y,z)$

$f : \{\mathbf{T},\mathbf{F}\} \times \{\mathbf{T},\mathbf{F}\} \times \{\mathbf{T},\mathbf{F}\} \to \{\mathbf{T},\mathbf{F}\}$
or also $\quad f : \{\mathbf{T},\mathbf{F}\}^3 \to \{\mathbf{T},\mathbf{F}\}$

- f is a function which returns **T** iff exactly two of its three variables are assigned the value **T**
- there are many different ways of writing a formula for f
- here, we want it as a **propositional wff**
- there are many different ways of writing it as a **propositional wff**
- here, we want it as a propositional wff in DNF (**disjunctive normal form**)
- we also want it as a propositional wff in CNF (**conjunctive normal form**)

from *truth tables* to *propositional formulas in DNF*

x	y	z	$f(x,y,z)$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

[1] In fact, this is a CDNF (*canonical DNF*) because it is a disjunction of *minterms* – look up the definition on the Web.

from *truth tables* to *propositional formulas in DNF*

x	y	z	$f(x, y, z)$
T	**T**	**T**	**F**
T	**T**	**F**	**T**
T	**F**	**T**	**T**
T	**F**	**F**	**F**
F	**T**	**T**	**T**
F	**T**	**F**	**F**
F	**F**	**T**	**F**
F	**F**	**F**	**F**

- read **T** as " *true* " and **F** as " *false* "
- f is *true* in **row 2**, in **row 3**, in **row 5**
- writing f as a propositional wff φ in DNF:[1]

(**row 2** true) or (**row 3** true) or (**row 5** true)

$$\varphi \triangleq (x \wedge y \wedge \neg z) \vee (x \wedge \neg y \wedge z) \vee (\neg x \wedge y \wedge z)$$

[1] In fact, this is a CDNF (*canonical DNF*) because it is a disjunction of *minterms* – look up the definition on the Web.

from *truth tables* to *propositional formulas in CNF* ???

- several ways of doing this
- one way: consider the "negation of f" – call it g

x	y	z	$g(x,y,z)$
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

from *truth tables* to *propositional formulas in CNF* ???

- several ways of doing this
- one way: consider the "negation of f" – call it g

x	y	z	$g(x,y,z)$
T	**T**	**T**	**T**
T	**T**	**F**	**F**
T	**F**	**T**	**F**
T	**F**	**F**	**T**
F	**T**	**T**	**F**
F	**T**	**F**	**T**
F	**F**	**T**	**T**
F	**F**	**F**	**T**

- writing g as a propositional wff ψ in DNF:

(**row 1** true) or (**row 4** true) or (**row 6** true) or (**row 7** true) or (**row 8** true)

$$\psi \triangleq (x \wedge y \wedge z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge y \wedge \neg z) \vee (\neg x \wedge \neg y \wedge z) \vee (\neg x \wedge \neg y \wedge \neg z)$$

from *truth tables* to *propositional formulas in CNF* ???

recall de Morgan's laws:

$$\neg(p \wedge q) \equiv (\neg p \vee \neg q)$$
$$\neg(p \vee q) \equiv (\neg p \wedge \neg q)$$

from *truth tables* to *propositional formulas in CNF* ???

recall de Morgan's laws:

$$\begin{aligned} \neg(p \wedge q) &\equiv (\neg p \vee \neg q) \\ \neg(p \vee q) &\equiv (\neg p \wedge \neg q) \end{aligned}$$

apply de Morgan's laws to ψ:

$$\begin{aligned} \neg\psi &\equiv \neg(x \wedge y \wedge z) \wedge \\ &\quad\; \neg(x \wedge \neg y \wedge \neg z) \wedge \\ &\quad\; \neg(\neg x \wedge y \wedge \neg z) \wedge \\ &\quad\; \neg(\neg x \wedge \neg y \wedge z) \wedge \\ &\quad\; \neg(\neg x \wedge \neg y \wedge \neg z) \\ &\equiv (\neg x \vee \neg y \vee \neg z) \wedge \\ &\quad\; (\neg x \vee y \vee z) \wedge \\ &\quad\; (x \vee \neg y \vee z) \wedge \\ &\quad\; (x \vee y \vee \neg z) \wedge \\ &\quad\; (x \vee y \vee z) \qquad\qquad \equiv \varphi \quad \text{(in CNF)} \end{aligned}$$

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