CS 511, Fall 2018, Handout 01 Propositional Logic:

From Truth Tables To Conjunctive Normal Forms (CNF) and Disjunctive Normal Forms (DNF)

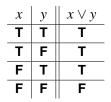
Assaf Kfoury

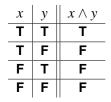
4 September 2018 (adjusted 11 September 2018)

Assaf Kfoury, CS 511, Fall 2018, Handout 01

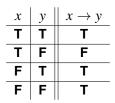
# some familiar truth-tables:

logical "or" ( $\lor$ ) and logical "and" ( $\land$ )



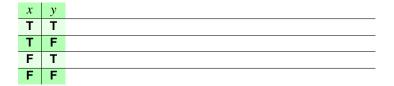


logical "implication"  $(\rightarrow)$ 



and similarly for "negation"  $(\neg)$  and many other logical connectives . . . .

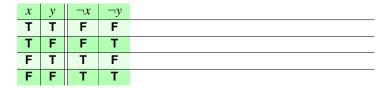
- start with all the propositional atoms in the wff  $\varphi$
- incrementally, consider each sub-wff of  $\varphi$ , from innermost to outermost



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x	У	$\neg x$
Т	Т	F
Т	F	F
F	Т	Т
F	F	Т

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x	y	$\neg x$	$\neg y$	$x \rightarrow \neg y$	
Т	Т	F	F	F	
Т	F	F	Т	Т	
F	Т	Т	F	Т	
F	F	Т	Т	Т	

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x	y	$\neg x$	$\neg y$	$x \rightarrow \neg y$	$y \lor \neg x$	
Т	Т	F	F	F	Т	
Т	F	F	Т	Т	F	
F	Т	Т	F	Т	Т	
F	F	Т	Т	Т	Т	

- start with all the propositional atoms in the wff  $\varphi$
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x	У	$\neg x$	$\neg y$	$x \rightarrow \neg y$	$y \lor \neg x$	$(x \to \neg y) \to (y \lor \neg x)$
Т	Т	F	F	F	Т	Т
Т	F	F	Т	Т	F	F
F	Т	Т	F	Т	Т	Т
F	F	Т	Т	Т	Т	Т

- start with all the propositional atoms in the wff  $\varphi$
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Т	Т	F	F	F	Т	Т
Т	F	F	Т	Т	F	F
F	Т	Т	F	Т	Т	Т
F	F	Т	Т	Т	Т	Т

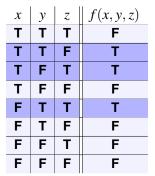
- ▶ propositional wff  $\varphi$  is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes  $\varphi$  true.
- Propositional wff  $\varphi$  is a **tautology** if **every** assignment of truth-values to the propositional atoms makes  $\varphi$  true.

- start with all the propositional atoms in the wff  $\varphi$
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x	У	$\neg x$	$\neg y$	$x \rightarrow \neg y$	$y \lor \neg x$	$(x \to \neg y) \to (y \lor \neg x)$
Т	Т	F	F	F	Т	Т
Т	F	F	Т	Т	F	F
F	Т	Т	F	Т	Т	Т
F	F	Т	Т	Т	Т	Т

- ▶ propositional wff  $\varphi$  is **satisfiable** if **there is** an assignment of truth-values to the propositional atoms which makes  $\varphi$  true.
- Propositional wff  $\varphi$  is a **tautology** if **every** assignment of truth-values to the propositional atoms makes  $\varphi$  true.
- ▶  $\varphi \triangleq (x \to \neg y) \to (y \lor \neg x)$  is satisfiable, but is not a tautology.

# example of a truth-table



input variables:  $\{x, y, z\}$ 

output: f(x, y, z)

```
\begin{split} f: \{\mathbf{T}, \mathbf{F}\} \times \{\mathbf{T}, \mathbf{F}\} \times \{\mathbf{T}, \mathbf{F}\} & \rightarrow \{\mathbf{T}, \mathbf{F}\} \\ \text{or also} \quad f: \{\mathbf{T}, \mathbf{F}\}^3 \rightarrow \{\mathbf{T}, \mathbf{F}\} \end{split}
```

# example of a truth-table

x	y y	z	$\int f(x,y,z)$
Т	Т	Т	F
Т	Т	F	Т
Т	F	Т	Т
Т	F	F	F
F	Т	Т	Т
F	Т	F	F
F	F	Т	F
F	F	F	F

input variables:  $\{x, y, z\}$ 

output: f(x, y, z)

```
\begin{split} f: \{\mathbf{T},\!\mathbf{F}\}\!\times\!\!\{\mathbf{T},\!\mathbf{F}\}\!\times\!\!\{\mathbf{T},\!\mathbf{F}\} &\rightarrow \{\mathbf{T},\!\mathbf{F}\}\\ \text{or also} \quad f: \{\mathbf{T},\!\mathbf{F}\}^3 \rightarrow \{\mathbf{T},\!\mathbf{F}\} \end{split}
```

- f is a function which returns T iff exactly two of its three variables are assigned the value T
- there are many different ways of writing a formula for f

### example of a truth-table

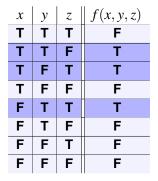
x	y y	z	$\int f(x,y,z)$
Т	Т	Т	F
Т	Т	F	Т
Т	F	Т	Т
Т	F	F	F
F	Т	Т	Т
F	Т	F	F
F	F	Т	F
F	F	F	F

input variables:  $\{x, y, z\}$ 

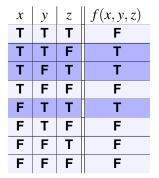
output: f(x, y, z)

$$\begin{split} f: \{\mathbf{T}, \mathbf{F}\} \times \{\mathbf{T}, \mathbf{F}\} & \to \{\mathbf{T}, \mathbf{F}\} \\ \text{or also} \quad f: \{\mathbf{T}, \mathbf{F}\}^3 \to \{\mathbf{T}, \mathbf{F}\} \end{split}$$

- f is a function which returns T iff exactly two of its three variables are assigned the value T
- there are many different ways of writing a formula for f
- here, we want it as a propositional wff
- there are many different ways of writing it as a propositional wff
- here, we want it as a propositional wff in DNF (disjunctive normal form)
- we also want it as a propositional wff in CNF (conjunctive normal form)



<sup>&</sup>lt;sup>1</sup> In fact, this is a CDNF (*canonical DNF*) because it is a disjunction of *minterms* – look up the definition on the Web.



read T as " true " and F as " false "

f is true in row 2, in row 3, in row 5

• writing f as a propositional wff  $\varphi$  in DNF:<sup>1</sup>

(row 2 true) or (row 3 true) or (row 5 true)  $\varphi \triangleq (x \land y \land \neg z) \lor (x \land \neg y \land z) \lor (\neg x \land y \land z)$ 

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- several ways of doing this
- one way: consider the "negation of f" call it g

x	y y	z	g(x, y, z)
Т	Т	Т	Т
Т	Т	F	F
Т	F	Т	F
Т	F	F	Т
F	Т	Т	F
F	Т	F	Т
F	F	Т	Т
F	F	F	Т

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  - one way: consider the "negation of f" call it g

x	y y	z	g(x, y, z)
Т	Т	Т	Т
Т	Т	F	F
Т	F	Т	F
Т	F	F	Т
F	Т	Т	F
F	Т	F	Т
F	F	Т	Т
F	F	F	Т

• writing g as a propositional wff  $\psi$  in DNF:

(row 1 true) or (row 4 true) or (row 6 true) or (row 7 true) or (row 8 true)  $\psi \triangleq (x \land y \land z) \lor (x \land \neg y \land \neg z) \lor (\neg x \land y \land \neg z) \lor (\neg x \land \neg y \land z) \lor (\neg x \land \neg y \land \neg z)$ 

recall de Morgan's laws:

$$\neg (p \land q) \equiv (\neg p \lor \neg q)$$
  
$$\neg (p \lor q) \equiv (\neg p \land \neg q)$$

recall de Morgan's laws:

$$\begin{array}{l} \neg (p \wedge q) \ \equiv (\neg p \vee \neg q) \\ \neg (p \vee q) \ \equiv (\neg p \wedge \neg q) \end{array}$$

apply de Morgan's laws to  $\psi$ :

$$\neg \psi \equiv \neg (x \land y \land z) \land 
\neg (x \land \neg y \land \neg z) \land 
\neg (\neg x \land y \land \neg z) \land 
\neg (\neg x \land \neg y \land z) \land 
\neg (\neg x \land \neg y \land \neg z) 
\equiv (\neg x \lor \neg y \lor \neg z) \land 
(\neg x \lor y \lor z) \land 
(x \lor y \lor z) \land$$

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