CS 511, Fall 2018, Handout 07

Semantics of *Classical* Propositional Logic

(Continued)

Soundness, Completeness, Compactness

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Soundness

- Let Γ a (possibly infinite) set of propositional wff's.

  If, for every model/interpretation/valuation
  (*i.e.*, assignment of truth values to prop atoms), it holds that:

  - whenever all the wff's in Γ evaluate to **T**,
  - it is also the case that ψ evaluates to **T**,

  then we write:

  $\Gamma \models \psi$ in words, "Γ semantically entails ψ"

- **Theorem (Soundness):** If $\Gamma \vdash \psi$ then $\Gamma \models \psi$.

  (Slightly stronger than the statement of Soundness in [LCS, Theorem 1.35, p 46]:
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.)

- **Proof idea:** "Course-of-values" induction on $n \geqslant 1$
  (in a later **Handout 08**).

Completeness

- **Theorem (Completeness):** If $\Gamma \models \psi$ then $\Gamma \vdash \psi$.

  (Stronger than the statement of Completeness in [LCS, Corollary 1.39, p 53]:
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$.)

- **Proof idea in [LCS]** (which works if Γ is a finite set $\{\varphi_1, \ldots, \varphi_n\}$):

  Establish 3 preliminary results. From $\varphi_1, \ldots, \varphi_n \models \psi$, show that:

  1. $\models \varphi_1 \to (\varphi_2 \to (\varphi_3 \to (\cdots(\varphi_n \to \psi)\cdots)))$ holds.
  2. $\vdash \varphi_1 \to (\varphi_2 \to (\varphi_3 \to (\cdots(\varphi_n \to \psi)\cdots)))$ is a valid sequent.
  3. $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ is a valid sequent.

- If Γ is infinite, we need another preliminary result: Compactness.

Compactness

- Γ is said to be **satisfiable** if there is a model/interpretation/valuation which satisfies/makes true every φ in Γ.
- **Theorem (Compactness)** (not in [LCS]):
  Γ is satisfiable iff every finite subset of Γ is satisfiable.
- **Corollary** (not in [LCS]):
  If $\Gamma \models \psi$ then there is a finite subset $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \models \psi$.
- For proofs of Compactness above and its corollary, click here.

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