CS 511, Fall 2018, Handout 08

Classical Propositional Logic: Proof Sketch of its Soundness

Assaf Kfoury

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Soundness once more

- **Theorem (Soundness):**
  If $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi$ then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.
- **Proof: Course-of-values induction**
  (sometimes called **strong induction**) on $k \geqslant 1$,
  where k is number of lines in a formal proof.
- Base step: Consider $k = 1$ (quite trivial!).
  In this case $n = k = 1$.

  From a given sequent $\varphi_1 \vdash \psi$, we want to show $\varphi_1 \models \psi$.

  Such a sequent implies $\varphi_1 = \psi$, i.e., $\psi \vdash \psi$.

  Hence, $\psi \models \psi$,

  which is the same as $\varphi_1 \models \psi$. (QED for base case)
- Inductive step: Consider arbitrary $k \geqslant 2$.
  (Actually for $1 \leqslant k \leqslant n$, it is trivial again. Interesting case: $k > n$.)

  Induction hypothesis (IH): Soundness holds for every $k' < k$.

Structure of a formal proof with n premises:

1	φ_1	premise
2	φ_2	premise
$\vdots$	$\vdots$	
n	φ_n	premise
$\vdots$	$\vdots$	
k	ψ	justification

- Last line in the proof, line k, is the result of 1 or 2 or . . . preceding it.
- Consider each possible "justification" separately: finitely many.
- Suppose "justification" is "$\wedge$i".
  This means line k uses lines k_1 and k_2, with $k_1, k_2 < k$.
- Use IH on $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_1}$ and $\varphi_1, \varphi_2, \ldots, \varphi_n \vdash \psi_{k_2}$.

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