# CS 511, Fall 2018, Handout 15

# Syntax of Predicate Logic (aka First-Order Logic)

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# from English reasoning to formal reasoning:

**for all** $x$, **if** $x$ is a bird **then** $x$ has wings

**for all** $x$, **if** $x$ has wings **then** $x$ can fly

**Coco** is a bird

**Coco** has wings

**Coco**'s *mother* can fly

# from English reasoning to formal reasoning:

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| **for all** $x$, **if** $x$ is a bird **then** $x$ has wings | $\forall x\ (B(x)\ \rightarrow\ W(x))$ |
| **for all** $x$, **if** $x$ has wings **then** $x$ can fly | $\forall x\ (W(x)\ \rightarrow\ F(x))$ |
| **Coco** is a bird | $B(\mathbf{C})$ |
| **Coco** has wings | $W(\mathbf{C})$ |
| **Coco**'s *mother* can fly | $F(m(\mathbf{C}))$ |

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| it is **not** the case that **for all** $x$ ... | $\neg(\forall x\,(B(x) \;\rightarrow\; W(x)))$ |
| **there exists** an $x$ such that ... | $\exists x\,(B(x) \;\wedge\; \neg W(x))$ |

# WFF's of predicate logic

- *vocabulary* (a.k.a. *similarity type*, a.k.a. *signature*):

- *terms*:

- *well-formed formulas*:

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  set $\mathcal{P}$ of **predicate** symbols, each of arity $n \geqslant 0$
  set $\mathcal{F}$ of **function** symbols, each of arity $n \geqslant 1$
  set $\mathcal{C}$ of **constant** symbols, (a.k.a. functions of arity $= 0$)

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- *terms*:

  a variable $x$ is a term
  a constant $c \in \mathcal{C}$ is a term
  if $t_1, \ldots, t_n$ are terms and $f \in \mathcal{F}$ is $n$-ary, $f(t_1, \ldots, t_n)$ is a term

  - as a BNF definition:

  $$t \ ::= \ x \mid c \mid f(t, \ldots, t)$$

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  then $P(t_1, \ldots, t_n)$ is a WFF (a.k.a. **atomic** WFF)

  if $t_1, t_2$ are terms,
  then $t_1 \doteq t_2$ is a WFF (a.k.a. **atomic** WFF)

  if $\varphi$ is a WFF, then so is $\neg\varphi$

  if $\varphi$ and $\psi$ are WFF's,
  then so are $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, and $(\varphi \to \psi)$

  if $\varphi$ is a WFF and $x$ is a variable,
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- are all WFF's of propositional logic WFF's of predicate logic ???

# **free** and **bound** variables

- a variable $x$ may occur **free** or **bound** in a WFF $\varphi$
- if $x$ is bound in $\varphi$, then there are $\geqslant 0$ **bound** occurrences of $x$ and $\geqslant 1$ **binding** occurrences of $x$ in $\varphi$
- a **binding** occurrence of $x$ is of the form "$\forall x$" or "$\exists x$"
- if a binding occurrence of $x$ occurs as $(\mathbf{Q}\, x\, \varphi)$ where $\mathbf{Q} \in \{\forall, \exists\}$, then $\varphi$ is the **scope** of the binding occurrence
- scopes of two binding occurrences "$\mathbf{Q}\, x$" and "$\mathbf{Q}'\, x'$" may be

**disjoint**: $\quad \cdots\ (\mathbf{Q}\, x \underbrace{\cdots\ \cdots}) \ \cdots\ (\mathbf{Q}'\, x' \underbrace{\cdots\ \cdots}) \ \cdots$

or **nested**: $\quad \cdots\ (\mathbf{Q}\, x \underbrace{\cdots\ (\mathbf{Q}'\, x' \underbrace{\cdots\ \cdots})\ \cdots})\ \cdots$

but cannot **overlap**

# **free** and **bound** variables (continued)

- the set of **free variables** in terms $t$ and WFF's $\varphi$:

$$\mathsf{FV}(t) = \begin{cases} \varnothing & \text{if } t = c \\ \{x\} & \text{if } t = x \\ \mathsf{FV}(t_1) \cup \cdots \cup \mathsf{FV}(t_n) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

$$\mathsf{FV}(\varphi) = \begin{cases} \mathsf{FV}(t_1) \cup \cdots \cup \mathsf{FV}(t_n) & \text{if } \varphi = P(t_1, \ldots, t_n) \\ \mathsf{FV}(t_1) \cup \mathsf{FV}(t_2) & \text{if } \varphi = (t_1 \doteq t_2) \\ \mathsf{FV}(\varphi') & \text{if } \varphi = \neg\varphi' \\ \mathsf{FV}(\varphi_1) \cup \mathsf{FV}(\varphi_2) & \text{if } \varphi = (\varphi_1 \star \varphi_2), \star \in \{\wedge, \vee, \to\} \\ \mathsf{FV}(\varphi') - \{x\} & \text{if } \varphi = (\mathbf{Q}x\ \varphi') \text{ and } \mathbf{Q} \in \{\forall, \exists\} \end{cases}$$

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- **assumption**: every variable $x$ has $\leqslant 1$ binding occurrence in any WFF (is this realistic?)

  this assumption is not essential, but without it, a variable may occur both free and bound in the same WFF.

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- how to satisfy the following **assumption**:
  every variable $x$ has $\leqslant 1$ binding occurrence in any WFF?
- consider a WFF $\varphi$ (**not** satisfying the **assumption**), say:

$$\varphi = \quad \cdots \Big(\mathbf{Q}_1\, x\, (\cdots\ x\ \cdots)\Big)\ \cdots\ \Big(\mathbf{Q}_2\, x\, (\cdots\ x\ \cdots)\Big)\ \cdots$$

  where $\mathbf{Q}_1, \mathbf{Q}_2 \in \{\forall, \exists\}$

# **free** and **bound** variables (continued)

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- how to satisfy the following **assumption**:
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- consider a WFF $\varphi$ (**not** satisfying the **assumption**), say:

  $$\varphi = \quad \cdots \Big( \mathbf{Q}_1\, x \,(\cdots\; x\; \cdots) \Big) \;\cdots\; \Big( \mathbf{Q}_2\, x \,(\cdots\; x\; \cdots) \Big) \;\cdots$$

  where $\mathbf{Q}_1, \mathbf{Q}_2 \in \{\forall, \exists\}$
- is $\varphi$ equivalent to:

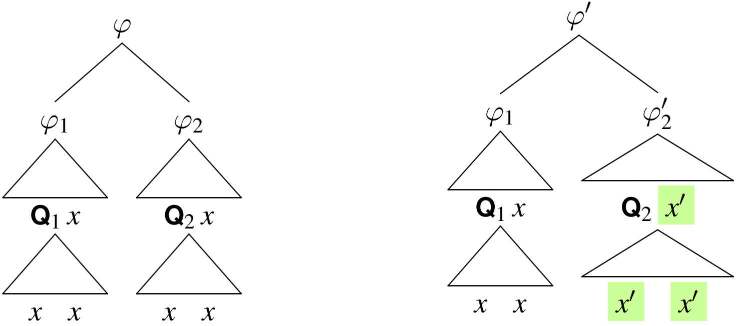
  $$\varphi' = \quad \cdots \Big( \mathbf{Q}_1\, x \,(\cdots\; x\; \cdots) \Big) \;\cdots\; \Big( \mathbf{Q}_2\; x' \;(\cdots\; x'\; \cdots)\; \cdots \Big) \;\mathbf{??}$$

- yes, $\varphi$ and $\varphi'$ are equivalent

  **Exercise:** define the algorithm to transform $\varphi$ into $\varphi'$

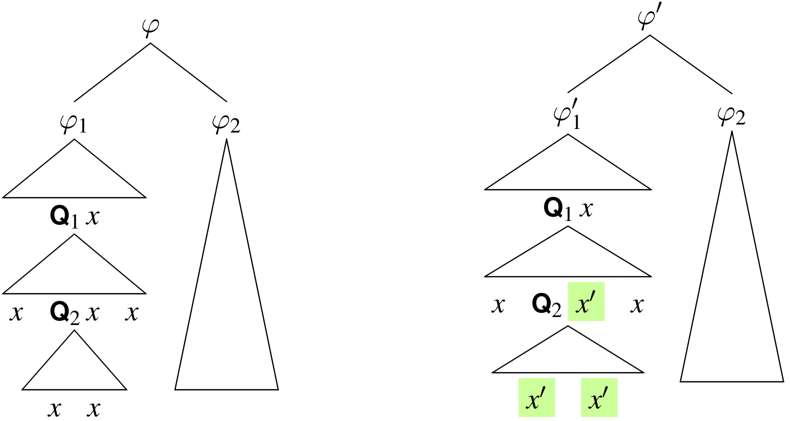
# **free** and **bound** variables (continued)

renaming binding occurrences "$\mathbf{Q}_1\, x$" and "$\mathbf{Q}_2\, x$" in **disjoint** scopes



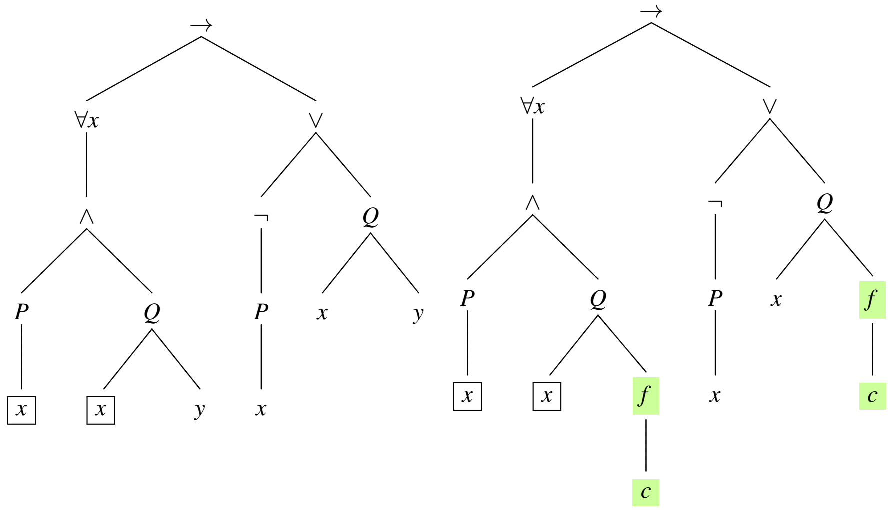
# **free** and **bound** variables (continued)

renaming binding occurrences "$\mathbf{Q}_1\, x$" and "$\mathbf{Q}_2\, x$" in **nested** scopes



# substitution examples in $\varphi = (\forall x\ (P(x) \wedge Q(x,y))) \rightarrow (\neg P(x) \vee Q(x,y))$

substitute $f(c)$ for $y$ in $\varphi$: $\varphi[f(c)/y]$ (also written $\varphi[y := f(c)]$)

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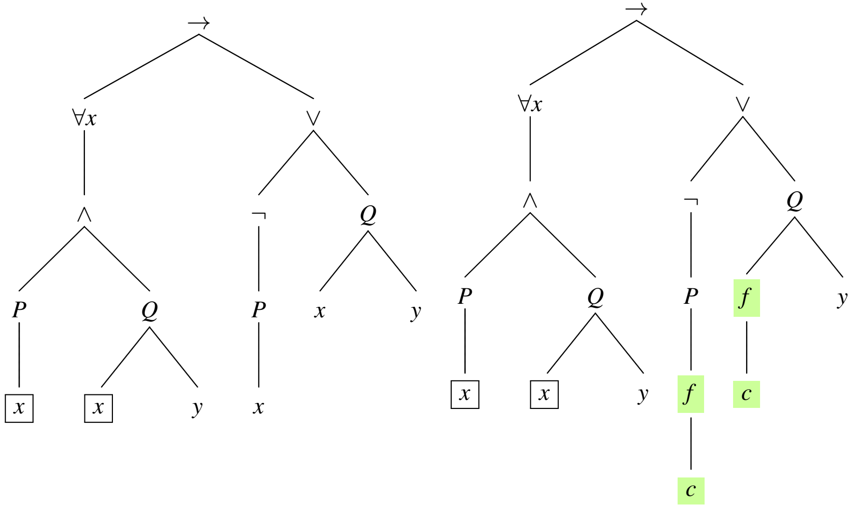


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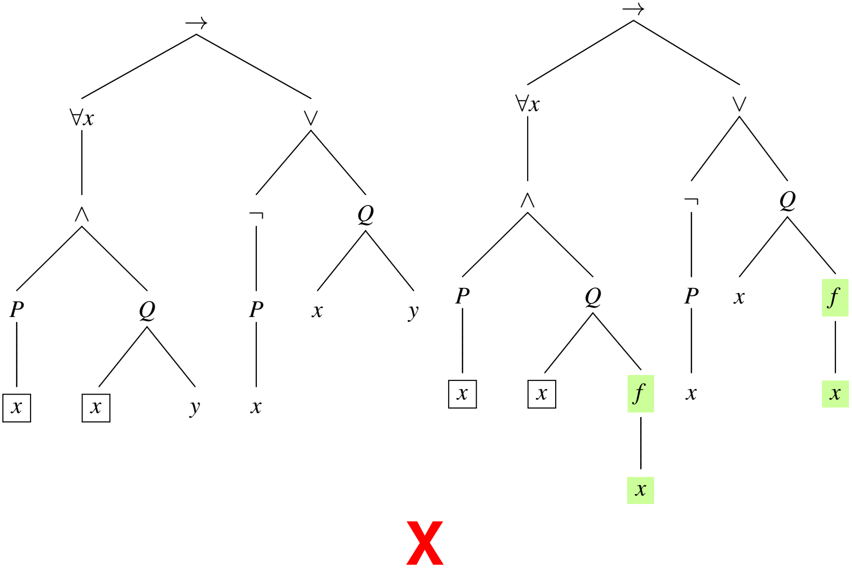

# formal definition of substitution

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- given: term $t$, WFF $\varphi$, variable $x$, term $u$

$$t[u/x] = \begin{cases} c & \text{if } t = c \\ u & \text{if } t = x \\ y & \text{if } t = y \text{ and } y \neq x \\ f(t_1[u/x], \ldots, t_n[u/x]) & \text{if } t = f(t_1, \ldots, t_n) \end{cases}$$

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$$\varphi[u/x] = \begin{cases} P(t_1[u/x], \ldots, t_n[u/x]) & \text{if } \varphi = P(t_1, \ldots, t_n) \\ (t_1[u/x] \doteq t_2[u/x]) & \text{if } \varphi = (t_2 \doteq t_2) \\ \neg(\varphi'[u/x]) & \text{if } \varphi = \neg\varphi' \\ \varphi_1[u/x] \star \varphi_2[u/x] & \text{if } \varphi = \varphi_1 \star \varphi_2 \text{ and} \\ & \star \in \{\wedge, \vee, \rightarrow\} \\ \mathbf{Q}y\,(\varphi'[u/x]) & \text{if } \varphi = \mathbf{Q}y\,\varphi', \\ & \mathbf{Q} \in \{\forall, \exists\}, x \neq y, \text{ and} \\ & u \text{ is } \textbf{substitutable} \text{ for } x \text{ in } \varphi \\ \mathbf{Q}y\,\varphi' & \text{if } \varphi = \mathbf{Q}y\,\varphi', \\ & \mathbf{Q} \in \{\forall, \exists\}, x = y \end{cases}$$

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