CS 511, Fall 2018, Handout 16

Predicate Logic: Proof Rules of Natural Deduction

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proof rules for equality

- equality **introduction**

$$\frac{}{t \doteq t} \doteq\mathrm{i}$$

- equality **elimination**

$$\frac{t_1 \doteq t_2 \qquad \varphi[t_1/x]}{\varphi[t_2/x]} \doteq\mathrm{e}$$

formal proof for "$\doteq$" is symmetric

1	$u_1 \doteq u_2$	premise
2	$u_1 \doteq u_1$	$\doteq$i
3	$u_2 \doteq u_1$	$\doteq$e $1, 2$

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2	$u_1 \doteq u_1$	$\doteq$i
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Question: What above corresponds to the WFF φ in the rule $\doteq$e?

Answer: "$x \doteq u_1$" corresponds to φ in the rule $\doteq$e, so that "$u_1 \doteq u_1$" corresponds to $\varphi[u_1/x]$ & "$u_2 \doteq u_1$" corresponds to $\varphi[u_2/x]$

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We have formally proved
$u_1 \doteq u_2 \vdash u_2 \doteq u_1$

and we can therefore use as a **derived proof rule**

$$\frac{t_1 \doteq t_2}{t_2 \doteq t_1} \quad \doteq \text{symmetric}$$

formal proof for "$\doteq$" is transitive

1	$u_2 \doteq u_3$	premise
2	$u_1 \doteq u_2$	premise
3	$u_1 \doteq u_3$	$\doteq$e $1, 2$

formal proof for "$\doteq$" is transitive

1	$u_2 \doteq u_3$	premise
2	$u_1 \doteq u_2$	premise
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Answer: "$u_1 \doteq x$" corresponds to φ in the rule $\doteq$e, so that "$u_1 \doteq u_3$" corresponds to $\varphi[u_3/x]$ & "$u_1 \doteq u_2$" corresponds to $\varphi[u_2/x]$

formal proof for "$\doteq$" is transitive

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We have formally proved

$$u_1 \doteq u_2,\ u_2 \doteq u_3 \ \vdash\ u_1 \doteq u_3$$

and we can therefore use as a **derived proof rule**

$$\frac{t_1 \doteq t_2 \qquad t_2 \doteq t_3}{t_1 \doteq t_3} \ \doteq \text{transitive}$$

proof rules for universal quantification

- universal quantifier **elimination**

$$\frac{\forall x\,\varphi}{\varphi[t/x]}\ \forall x\ \text{e}$$

(usual assumption: t is substitutable for x)

- universal quantifier **introduction**

$$\frac{\boxed{\begin{array}{lcr} x_0 & & \text{fresh}\\ & \vdots & \\ & \varphi[x_0/x] & \end{array}}}{\forall x\,\varphi}\ \forall x\ \text{i}$$

proof rules for existential quantification

- existential quantifier **introduction**

$$\frac{\varphi[t/x]}{\exists x\, \varphi}\ \exists x \text{ i}$$

- existential quantifier **elimination**

$$\frac{\exists x\, \varphi \qquad \boxed{\begin{array}{lll} x_0 & & \text{fresh} \\ & \varphi[x_0/x] & \text{assumption} \\ & \vdots & \\ & \chi & \end{array}}}{\chi}\ \exists x \text{ e}$$

(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

proof rules for existential quantification

- existential quantifier **introduction**

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- existential quantifier **elimination**

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(x_0 cannot occur outside its box, in particular, it cannot occur in χ)

- **Note carefully**:
  Rule $(\exists x\ \mathsf{e})$ introduces both a **fresh** variable and an **assumption**.

example: $\forall x \, \forall y \, \varphi(x,y) \;\vdash\; \forall y \, \forall x \, \varphi(x,y)$

	1	$\forall x \, \forall y \, \varphi(x,y)$	premise
y_0	2		fresh y_0
x_0	3		fresh x_0
	4	$\forall y \, \varphi(x_0,y)$	$\forall x$ e, 1
	5	$\varphi(x_0,y_0)$	$\forall x$ e, 4
	6	$\forall x \, \varphi(x,y_0)$	$\forall x$ i, 5
	7	$\forall y \, \forall x \, \varphi(x,y)$	$\forall y$ i, 6

example: $\forall x\,(P(x) \to Q(x)),\ \forall x\,P(x) \ \vdash\ \forall x\,Q(x)$

	1	$\forall x\,(P(x) \to Q(x))$	premise
	2	$\forall x\,P(x)$	premise
x_0	3		fresh x_0
	4	$P(x_0) \to Q(x_0)$	$\forall x$ e, 1
	5	$P(x_0)$	$\forall x$ e, 2
	6	$Q(x_0)$	$\to$e, $4, 5$
	7	$\forall x\,Q(x)$	$\forall x$ i, 3-6

example: $\exists x\,(\varphi(x) \vee \psi(x)) \;\vdash\; \exists x\,\varphi(x) \;\vee\; \exists x\,\psi(x)$

	1	$\exists x\,(\varphi(x) \vee \psi(x))$		premise
x_0	2			fresh x_0
	3	$\varphi(x_0) \vee \psi(x_0)$		assumption
	4	$\varphi(x_0)$	$\psi(x_0)$	assumption
	5	$\exists x\,\varphi(x)$	$\exists x\,\psi(x)$	$\exists x$ i, 4
	6	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$	$\vee$i, 5
	7	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\vee$e, 3, 4-6
	8	$\exists x\,\varphi(x) \vee \exists x\,\psi(x)$		$\exists x$ e, 1, 2-7

example: $\exists x\, \varphi(x) \ \vee\ \exists x\, \psi(x) \ \vdash\ \exists x\, (\varphi(x) \vee \psi(x))$

example: $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\vdash\; \exists x\, (\varphi(x) \vee \psi(x))$

- Yes, this is a derivable sequent – left to you.
- Hence, $\exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x) \;\dashv\vdash\; \exists x\, (\varphi(x) \vee \psi(x))$

example: $\exists x\, (\varphi(x) \wedge \psi(x)) \;\vdash\; \exists x\, \varphi(x) \;\wedge\; \exists x\, \psi(x)$

- Yes, this is a derivable sequent – similar to the formal proof of $\exists x\, (\varphi(x) \vee \psi(x)) \;\vdash\; \exists x\, \varphi(x) \;\vee\; \exists x\, \psi(x)$

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- **Question:** $\boldsymbol{\exists x\,\varphi(x) \;\wedge\; \exists x\,\psi(x) \;\vdash\; \exists x\,(\varphi(x) \wedge \psi(x))}$ **??**
  No, this is not a derivable sequent

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  No, this is not a derivable sequent

  Find an interpretation (a "model") where

  $\exists x\, \varphi(x) \;\wedge\; \exists x\, \psi(x)$ is **true**, but

  $\exists x\, (\varphi(x) \wedge \psi(x))$ is **false**

- Hence, $\exists x\, (\varphi(x) \wedge \psi(x)) \;\not\dashv\vdash\; \exists x\, \varphi(x) \;\wedge\; \exists x\, \psi(x)$

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REMEMBER! To show that a WFF is **NOT** derivable, it is generally easier to find an interpretation where the WFF is not satisfiable.

example: $\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \;\vdash\; \forall y\, Q(y)$

example: $\exists x\, P(x), \forall x\, \forall y\, (P(x) \to Q(y)) \vdash \forall y\, Q(y)$

	1	$\exists x\, P(x)$	premise
	2	$\forall x\, \forall y\, (P(x) \to Q(y))$	premise
y_0	3		fresh y_0
x_0	4		fresh x_0
	5	$P(x_0)$	assumption
	6	$\forall y\, (P(x_0) \to Q(y))$	$\forall x$ e, 2
	7	$P(x_0) \to Q(y_0)$	$\forall y$ e, 6
	8	$Q(y_0)$	$\to$e, 5, 7
	9	$Q(y_0)$	$\exists x$ e, 1, 4-8
	10	$\forall y\, Q(y)$	$\forall y$ i, 3-9

quantifier equivalences

Theorem

- $$\neg\forall x\,\varphi \;\dashv\vdash\; \exists x\,\neg\varphi$$
  $$\neg\exists x\,\varphi \;\dashv\vdash\; \forall x\,\neg\varphi$$
- Assume x is not free in ψ:
  $$\forall x\,\varphi \wedge \psi \;\dashv\vdash\; \forall x\,(\varphi \wedge \psi)$$
  $$\forall x\,\varphi \vee \psi \;\dashv\vdash\; \forall x\,(\varphi \vee \psi)$$
  $$\exists x\,\varphi \wedge \psi \;\dashv\vdash\; \exists x\,(\varphi \wedge \psi)$$
  $$\exists x\,\varphi \vee \psi \;\dashv\vdash\; \exists x\,(\varphi \vee \psi)$$
  $$\forall x\,(\psi \to \varphi) \;\dashv\vdash\; \psi \to \forall x\,\varphi$$
  $$\exists x\,(\varphi \to \psi) \;\dashv\vdash\; \forall x\,\varphi \to \psi$$
  $$\forall x\,(\varphi \to \psi) \;\dashv\vdash\; \exists x\,\varphi \to \psi$$
  $$\exists x\,(\psi \to \varphi) \;\dashv\vdash\; \psi \to \exists x\,\varphi$$
- $$\forall x\,\varphi \wedge \forall x\,\psi \;\dashv\vdash\; \forall x\,(\varphi \wedge \psi)$$
  $$\exists x\,\varphi \vee \exists x\,\psi \;\dashv\vdash\; \exists x\,(\varphi \vee \psi)$$

proof of only one quantifier equivalence, others in the book

- $\neg\forall x\, \varphi \ \vdash\ \exists x\, \neg\varphi$

proof of only one quantifier equivalence, others in the book

- $\neg \forall x\, \varphi \ \vdash\ \exists x\, \neg\varphi$

	1	$\neg\forall x\, \varphi$	premise
	2	$\neg\exists x\, \neg\varphi$	assumption
x_0	3		fresh x_0
	4	$\neg\varphi[x_0/x]$	assumption
	5	$\exists x\, \neg\varphi$	$\exists x$ i, 4
	6	$\bot$	$\neg$e, 5, 2
	7	$\varphi[x_0/x]$	PBC, 4-6
	8	$\forall x\, \varphi$	$\forall x$ i, 3-7
	9	$\bot$	$\neg$e, 8, 1
	10	$\exists x\, \neg\varphi$	PBC, 2-9

three fundamental questions

- **Question**
  Given a WFF φ, can we automate the answer to the query "$\vdash \varphi$??"
- **Question**
  Given a WFF φ, can we automate the answer to the query "$\nvdash \varphi$??"
- **Question**
  Given a formal proof

  1. φ_1
  2. φ_2
  3. $\vdots$

  n. φ_n

  can we automate the verification of the proof?

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