CS 511, Fall 2018, Handout 20

Examples of Relational/Algebraic Structures: Posets, Lattices, Heyting Algebras, Boolean Algebras, and more

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Algebraic Structures: definitions and examples

- An **algebraic structure** $\mathcal{A}$, or just an **algebra** $\mathcal{A}$, is a set A, called the **carrier set** or **underlying set** of $\mathcal{A}$, with one or more operations on the carrier A. (Search the Web, **here** and **here**, for more details.)

- Examples of **algebraic structures**:

  - $(\mathbb{Z}, +, \cdot)$
    the set of integers with **binary** operations addition "$+$" and multiplication "$\cdot$",

  - $(\mathbb{N}, \text{succ}, \text{pred}, 0, 1)$
    the set of natural numbers with **unary** operations, "succ" and "pred", and **nullary** operations, "0" and "1",

  - $(T, \text{node}, \text{Lt}, \text{Rt})$ where T is the least set such that:

    $$T \supseteq \{a, b, c\} \cup \{\, \langle t_1\ t_2 \rangle \mid t_1, t_2 \in T \,\}$$

    with one **binary** operation "node" and two **unary** operations "Lt" and "Rt", defined by:

Algebraic Structures: definitions and examples

$$\text{node} : T \times T \to T \quad \text{such that} \quad \text{node}(t_1, t_2) = \langle t_1\, t_2 \rangle$$

$$\text{Lt} : T \to T \quad \text{such that} \quad \text{Lt}(t) = \begin{cases} t_1 & \text{if } t = \langle t_1\ t_2 \rangle, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

$$\text{Rt} : T \to T \quad \text{such that} \quad \text{Rt}(t) = \begin{cases} t_2 & \text{if } t = \langle t_1\ t_2 \rangle, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

- Sometimes an **algebraic structure** includes two (or more) carriers, together with operations between them, in which case we say the algebraic structure is **two-sorted** (or **multi-sorted**).
- Examples of **two-sorted algebraic structures**:
  - $(\mathbb{Z}, \mathbb{B}, \leqslant, +, \cdot)$ where $\mathbb{B} = \{\mathbf{F}, \mathbf{T}\}$ and $\leqslant : \mathbb{Z} \times \mathbb{Z} \to \mathbb{B}$.
  - $(T, \mathbb{N}, \text{node}, \text{Lt}, \text{Rt}, \big|\ \big|, \text{height})$ where T is defined on the previous slide, with $\big|\ \big| : T \to \mathbb{N}$ and $\text{height} : T \to \mathbb{N}$.

Algebraic Structures: definitions and examples

- For a two-sorted structure such as $\mathcal{M} \triangleq (T, \mathbb{N}, \text{node}, \text{Lt}, \text{Rt}, |\ |, \text{height})$, we need to introduce two unary relation symbols, say R_1 and R_2, whose interpretations are the domains T and $\mathbb{N}$:

$$R_1^{\mathcal{M}} = T \quad \text{and} \quad R_2^{\mathcal{M}} = \mathbb{N}$$

- $\mathcal{M}$ satisfies the first-order sentence:

$$\big(\forall x.\, R_1(x) \vee R_2(x)\big) \;\wedge\; \neg\big(\exists x.\, R_1(x) \wedge R_2(x)\big)$$

- To assert that an element of the first domain T satisfies a wff $\varphi(x)$ with one free variable x, we write:

$$\exists x.\, R_1(x) \wedge \varphi(x)$$

(The book [LCS], Chapter 2, does not deal with multi-sorted structures.)

Algebraic Structures: definitions and examples

- Sometimes in a **two-sorted algebraic structure**, such as $(\mathbb{Z}, \mathbb{B}, \leqslant, +, \cdot)$ with the Boolean carrier $\mathbb{B}$ one of the two sorts, we can omit $\mathbb{B}$ and simply write $(\mathbb{Z}, \leqslant, +, \cdot)$.
- This assumes that it is clear to the reader that "$\leqslant$" is a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{B}$, *i.e.*, "$\leqslant$" is a binary **relation** (rather than a binary **function** or **operation**). As a binary relation, we can write: $\leqslant \;\subseteq\; \mathbb{Z} \times \mathbb{Z}$.
- Strictly speaking, a structure such as $(\mathbb{Z}, \leqslant, +, \cdot)$, which now includes **operations** as well as **relations**, is called a **relational structure** rather than just an algebraic structure.
- But the transition from **algebraic structures** to more general **relational structures** is not demarcated sharply.
- In particular, if a struture $\mathcal{A}$ includes one or two relations with standard meanings (such as "$\leqslant$"), we can continue to call $\mathcal{A}$ an **algebraic structure**.

Posets: definitions and examples

- A **partially ordered set**, or **poset** for short, is a set P with a **partial ordering** $\trianglelefteq$ on P, *i.e.*, for all $a, b, c \in P$, the ordering $\trianglelefteq$ satisfies:

$a \trianglelefteq a$	"$\trianglelefteq$ is **reflexive**"
$(a \trianglelefteq b$ and $b \trianglelefteq a)$ imply $a = b$	"$\trianglelefteq$ is **anti-symmetric**"
$(a \trianglelefteq b$ and $b \trianglelefteq c)$ imply $a \trianglelefteq c$	"$\trianglelefteq$ is **transitive**"

  The ordering $\trianglelefteq$ is **total** if it also satisfies for all $a, b \in P$:

  $$(a \trianglelefteq b) \text{ or } (b \trianglelefteq a)$$

- Examples of **posets**:

  (1) $(2^A, \trianglelefteq)$ where A is a non-empty set and $\trianglelefteq$ is $\subseteq$,
  (2) $(\mathbb{N} - \{0\}, \trianglelefteq)$ where $m \trianglelefteq n$ iff "m divides n",
  (3) $(\mathbb{N}, \trianglelefteq)$ where $\trianglelefteq$ is the usual ordering $\leqslant$.

  In (1) and (2), $\trianglelefteq$ is **not total**; in (3), $\trianglelefteq$ is **total**.

Lattices: definitions and examples

- An **lattice** $\mathcal{L}$ is an algebraic structure $(L, \trianglelefteq, \vee, \wedge)$ where $\vee$ and $\wedge$ are **binary operations**, and $\trianglelefteq$ is a **binary relation**, such that:
  - $(L, \trianglelefteq)$ is a poset,
  - for all $a, b \in L$, the **least upper bound** of a and b in the ordering $\trianglelefteq$
    - exists,
    - is unique,
    - and is the result of the operation "$a \vee b$",
  - for all $a, b \in L$, the **greatest lower bound** of a and b in $\trianglelefteq$
    - exists,
    - is unique,
    - and is the result of the operation "$a \wedge b$".
- Examples of **lattices**:
  - $(2^A, \trianglelefteq, \vee, \wedge)$ where $\trianglelefteq$ is $\subseteq$, $\vee$ is $\cup$, $\wedge$ is $\cap$
  - $(\mathbb{N} - \{0\}, \trianglelefteq, \vee, \wedge)$
    where $m \trianglelefteq n$ iff "m divides n", $\vee$ is "lcm", $\wedge$ is "gcd"

Distributive Lattices: definitions and examples

- A lattice $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge)$ is a **distributive lattice** if for all $a, b, c \in L$, the following **equations** – also called **axioms** or **equational axioms** – are satisfied:

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) \qquad \text{“}\wedge\text{” distributes over “}\vee\text{”}$$
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \qquad \text{“}\vee\text{” distributes over “}\wedge\text{”}$$

- Example of a **distributive lattice**:

$$(2^A, \subseteq, \cup, \cap)$$

- Is the following an example of a **distributive lattice**?

$$(\mathbb{N} - \{0\}, \text{“\_\_ divides \_\_”}, \text{lcm}, \text{gcd})$$

- For more details on **posets** and **lattices**, go to the Web: **here** (Hasse diagrams), **here** (distributive lattices), and **here**.

Bounded Lattices: definitions and examples

- A **bounded lattice** is an algebraic structure of the form

  $$\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \underset{\Uparrow}{\bot}, \underset{\Uparrow}{\top})$$

  where $\bot$ and $\top$ are **nullary** (or **0-ary**) **operations** on L (or, equivalently, **elements** in L) such that:

  1. $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge)$ is a lattice,
  2. $\bot \trianglelefteq a$ or, equivalently, $\bot \wedge a = \bot$ for every $a \in L$,
  3. $a \trianglelefteq \top$ or, equivalently, $a \vee \top = \top$ for every $a \in L$.

  The elements $\bot$ and $\top$ are uniquely defined. $\bot$ is the **minimum** element, and $\top$ is the **maximum** element, of the bounded lattice.

- Example of a **bounded lattice**: $(2^A, \subseteq, \cup, \cap, \underset{\Uparrow}{\varnothing}, \underset{\Uparrow}{A})$

- Example a **lattice** with a minimum, but **no** maximum:

  $$(\mathbb{N} - \{0\}, \text{“\_\_ divides \_\_”}, \mathrm{lcm}, \mathrm{gcd}, \underset{\Uparrow}{1})$$

Bounded Lattices: definitions and examples

- Let $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \perp, \top)$ be a **bounded lattice**. An element $a \in L$ has a **complement** $b \in L$ iff:

  $$a \wedge b = \perp \quad \text{and} \quad a \vee b = \top$$

  FACT: In a **bounded distributive lattice**, **complements** are uniquely defined,*i.e.*, an element $a \in L$ cannot have more than one complement $b \in L$.

  Proof. Exercise.

Complemented Lattices: definitions and examples

- A **complemented lattice** is a **bounded distributive lattice** $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top)$ where every element has a complement.
- Example of a **complemented lattice**: $(2^A, \subseteq, \cup, \cap, \varnothing, A)$
- Again, for more details various kinds of **lattices**, go to the Web: **here** (Hasse diagrams), **here** (distributive lattices), **here** (lattices).

Boolean Algebras: definitions and examples

- A **complemented lattice** $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top)$ is almost a **Boolean algebra**, but not quite!

  What is missing is an **additional operation** on L to map an element $a \in L$ to its **complement**.

- A <u>first definition</u> of a **Boolean algebra**:

$$\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top, \underset{\Uparrow}{\neg})$$

  where:

  1. $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top)$ is a **complemented lattice**,
  2. The new operation "$\neg$" is **unary** and maps every $a \in L$ to its complement, *i.e.*:

$$a \wedge (\neg a) = \bot \quad \text{and} \quad a \vee (\neg a) = \top$$

Boolean Algebras: definitions and examples

- A <u>second definition</u> of a **Boolean algebra** (easier to compare with Heyting algebras later) :

$$\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top, \underset{\Uparrow}{\to})$$

  where:

  1. $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top)$ is a **complemented lattice**,

  2. The new operation "$\to$" is **binary** such that $(a \to \bot)$ is the complement of a, for every every $a \in L$.

- **FACT**: The two preceding definitions of **Boolean algebras** are equivalent because we can define "$\to$" in terms of $\{\vee, \neg\}$:

$$a \to b \ := \ (\neg a) \vee b$$

  as well as define "$\neg$" in terms of $\{\to, \bot\}$:

$$\neg a \ := \ a \to \bot$$

Boolean Algebras: definitions and examples

- Examples of **Boolean algebras**:
  - For an arbitrary non-empty set A:

    $$(2^A, \subseteq, \cup, \cap, \varnothing, A, \overline{})$$

    where $\overline{X} = A - X$ for every $X \subseteq A$.
  - The standard 2-element Boolean algebra:

    $$(\{0,1\}, \leqslant, \vee, \wedge, 0, 1, \neg) \quad \text{or} \quad (\{0,1\}, \leqslant, \vee, \wedge, 0, 1, \to)$$

    where we write "0" for **F** and "1" for **T**.

Heyting Algebras: definitions and examples

- A **Heyting algebra** is an algebraic structure of the form

$$\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top, \underset{\Uparrow}{\rightarrow})$$

where:

- $\mathcal{L} = (L, \trianglelefteq, \vee, \wedge, \bot, \top)$ is a **bounded distributive lattice** – **not** necessarily a *complemented lattice*,
- The new operation "$\rightarrow$" is **binary** and satisfies the **equations**:

1. $a \rightarrow a = \top$
2. $a \wedge (a \rightarrow b) = a \wedge b$
3. $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$
4. $b \leqslant a \rightarrow b$

FACT: The preceding equations uniquely define the operation "$\rightarrow$". *Proof*. Exercise.

- **FACT:** Every Boolean algebra is a Heyting algebra. *Proof*. Exercise.

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