CS 511, Fall 2018, Handout 21

First-Order Logic: Soundness and Completeness

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consistency

 Γ is a set of WFF's.

- ightharpoonup Γ is **consistent** iff $\Gamma \not\vdash \bot$.
- ► **FACT.** The following three conditions are equivalent:
 - 1. Γ is consistent.
 - 2. For no WFF φ is it the case that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$.
 - 3. There is at least one WFF φ such that $\Gamma \not\vdash \varphi$.

consistency

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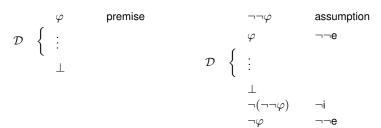
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- ► **FACT.** The following three conditions are equivalent:
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 - 2. For no WFF φ is it the case that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$.
 - 3. There is at least one WFF φ such that $\Gamma \not\vdash \varphi$.
- **Contrapositive FACT.** The following conditions are equivalent:
 - 4. Γ is inconsistent.
 - 5. There is a WFF φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$.
 - 6. For every WFF φ , it holds that $\Gamma \vdash \varphi$.
- Proof.
 - (4) \Rightarrow (6): Let $\Gamma \vdash \bot$. By the rule " \bot elimination", we add one more step in the proof to obtain $\Gamma \vdash \varphi$, which holds for every φ .
 - (6) \Rightarrow (5): Immediate.
 - (5) \Rightarrow (4): By the rule " \neg elimination", from the derivations $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$, we get $\Gamma \vdash \bot$.

consistency (continued)

Theorem. Let Γ be a set of WFF's and φ a WFF. We then have the two following (equivalent) statements:

- 1. $\Gamma \cup \{\varphi\}$ is inconsistent iff $\Gamma \vdash \neg \varphi$.
- 2. $\Gamma \cup \{\varphi\}$ is consistent iff $\Gamma \not\vdash \neg \varphi$.

Proof. It suffices to prove part 1 only. The simple right-to-left implication is left to you. For the left-to-right, suppose $\Gamma \cup \{\varphi\}$ is inconsistent. Hence we are given a formal derivation of the form on the left, and we build a new one on the right. The new one starts by opening a box with assumption $\neg\neg\varphi$, then uses rule " $\neg\neg e$ " and copies the given derivation with no change, and closes the initial box with rule " $\neg i$ ":



The new formal derivation on the right shows that $\Gamma \vdash \neg \varphi$ is a derivable sequent.

soundness

Theorem. Let Γ be a set of WFF's and φ a WFF.

If
$$\Gamma \vdash \varphi$$
 then $\Gamma \models \varphi$. (most common form for "soundness")

Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

soundness

Theorem. Let Γ be a set of WFF's and φ a WFF.

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$. (most common form for "soundness")

Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

Another form for "soundness" is the following:

Corollary. If Γ is satisfiable, then Γ is consistent.

Proof. Suppose Γ is inconsistent. Then there is a WFF φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg \varphi$, by part (5) on slide 3. By the previous theorem, both $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$, which is a contradiction.

completeness

One form of "completeness" is the following:

Theorem. Let Γ be a set of sentences (closed WFF's).

If Γ is consistent, then Γ is satisfiable.

Proof. By the Model-Existence Lemma (not in the book [LCS], and not included in these notes, look up "model-existence" lemma or theorem on the Web).

completeness

One form of "completeness" is the following:

Theorem. Let Γ be a set of sentences (closed WFF's).

If Γ is consistent, then Γ is satisfiable.

Proof. By the Model-Existence Lemma (not in the book [LCS], and not included in these notes, look up "model-existence" lemma or theorem on the Web).

Another form of "completeness", which is the most common:

Corollary.

If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Proof. Suppose $\Gamma \not\vdash \varphi$. Then $\Gamma \not\vdash \neg \neg \varphi$. So that $\Gamma \cup \{ \neg \varphi \}$ is consistent, by part 2 of theorem on slide 4. By the theorem on this slide, there is a model $\mathcal M$ of $\Gamma \cup \{ \neg \varphi \}$. Hence, $\mathcal M$ is a model of Γ but not of φ . Hence, $\Gamma \not\models \varphi$.

soundness and completeness – short form

For all WFF φ

 $\vdash \varphi \quad \text{if and only if} \quad \models \varphi$

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