CS 511, Fall 2018, Handout 21

First-Order Logic: Soundness and Completeness

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consistency

Γ is a set of WFF's.

- Γ is **consistent** iff $\Gamma \nvdash \bot$.
- **FACT.** The following three conditions are equivalent:
  1. Γ is consistent.
  2. For no WFF φ is it the case that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
  3. There is at least one WFF φ such that $\Gamma \nvdash \varphi$.

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- **Contrapositive FACT.** The following conditions are equivalent:
  4. Γ is inconsistent.
  5. There is a WFF φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$.
  6. For every WFF φ, it holds that $\Gamma \vdash \varphi$.
- **Proof.**
  (4) $\Rightarrow$ (6): Let $\Gamma \vdash \bot$. By the rule "$\bot$ elimination", we add one more step in the proof to obtain $\Gamma \vdash \varphi$, which holds for every φ.
  (6) $\Rightarrow$ (5): Immediate.
  (5) $\Rightarrow$ (4): By the rule "$\neg$ elimination", from the derivations $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, we get $\Gamma \vdash \bot$.

consistency (continued)

Theorem. Let Γ be a set of WFF's and φ a WFF.
We then have the two following (equivalent) statements:

1. $\Gamma \cup \{\varphi\}$ is inconsistent iff $\Gamma \vdash \neg\varphi$.
2. $\Gamma \cup \{\varphi\}$ is consistent iff $\Gamma \nvdash \neg\varphi$.

Proof. It suffices to prove part 1 only. The simple right-to-left implication is left to you. For the left-to-right, suppose $\Gamma \cup \{\varphi\}$ is inconsistent. Hence we are given a formal derivation of the form on the left, and we build a new one on the right. The new one starts by opening a box with assumption $\neg\neg\varphi$, then uses rule "$\neg\neg$e" and copies the given derivation with no change, and closes the initial box with rule "$\neg$i":

$$\begin{array}{lll} & \varphi & \text{premise} \\ \mathcal{D} \left\{ \begin{array}{c} \\ \vdots \\ \\ \end{array} \right. & & \\ & \bot & \end{array} \qquad\qquad \begin{array}{lll} & \neg\neg\varphi & \text{assumption} \\ & \varphi & \neg\neg\text{e} \\ \mathcal{D} \left\{ \begin{array}{c} \\ \vdots \\ \\ \end{array} \right. & & \\ & \bot & \\ & \neg(\neg\neg\varphi) & \neg\text{i} \\ & \neg\varphi & \neg\neg\text{e} \end{array}$$

The new formal derivation on the right shows that $\Gamma \vdash \neg\varphi$ is a derivable sequent.

soundness

Theorem. Let Γ be a set of WFF's and φ a WFF.

If $\Gamma \vdash \varphi$ then $\Gamma \models \varphi$. (most common form for "soundness")

Proof. Left to you. Consult also how each deduction rule is justified [LCS, Section 2.3].

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Another form for "soundness" is the following:

Corollary. If Γ is satisfiable, then Γ is consistent.

Proof. Suppose Γ is inconsistent. Then there is a WFF φ such that both $\Gamma \vdash \varphi$ and $\Gamma \vdash \neg\varphi$, by part (5) on slide 3. By the previous theorem, both $\Gamma \models \varphi$ and $\Gamma \models \neg\varphi$, which is a contradiction.

completeness

One form of "completeness" is the following:

Theorem. Let Γ be a set of sentences (closed WFF's).

If Γ is consistent, then Γ is satisfiable.

Proof. By the Model-Existence Lemma
(not in the book [LCS], and not included in these notes,
look up "model-existence" lemma or theorem on the Web).

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Theorem. Let Γ be a set of sentences (closed WFF's).

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Another form of "completeness", which is the most common:

Corollary.

If $\Gamma \models \varphi$ then $\Gamma \vdash \varphi$.

Proof. Suppose $\Gamma \nvdash \varphi$. Then $\Gamma \nvdash \neg\neg\varphi$. So that $\Gamma \cup \{\neg\varphi\}$ is consistent, by part 2 of theorem on slide 4. By the theorem on this slide, there is a model $\mathcal{M}$ of $\Gamma \cup \{\neg\varphi\}$. Hence, $\mathcal{M}$ is a model of Γ but not of φ. Hence, $\Gamma \not\models \varphi$.

soundness and completeness – short form

For all WFF φ

$$\boxed{\vdash \varphi \quad \text{if and only if} \quad \models \varphi}$$

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