CS 511, Fall 2018, Handout 23

Extended Example in First-Order Logic

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several structures over the domain $\mathbb{N}$ (assume "$\dot{-}$" is available)

structures over the domain of natural numbers	vocabulary/signature: predicate symbols	vocabulary/signature: function symbols
$\mathcal{N} \triangleq (\mathbb{N}, 0, S)$	$\mathscr{P} = \varnothing$	$\mathscr{F} = \{0, S\}$
$\mathcal{N}_1 \triangleq (\mathbb{N}, 0, S, <)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S\}$
$\mathcal{N}_2 \triangleq (\mathbb{N}, 0, S, <, +)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S, +\}$
$\mathcal{N}_3 \triangleq (\mathbb{N}, 0, S, <, +, \cdot)$	$\mathscr{P} = \{<\}$	$\mathscr{F} = \{0, S, +, \cdot\}$
$\mathcal{N}_4 \triangleq (\mathbb{N}, 0, S, <, +, \cdot, \mathsf{pr})$ $\mathsf{pr}(x) \triangleq$ true iff x is prime	$\mathscr{P} = \{<, \mathsf{pr}\}$	$\mathscr{F} = \{0, S, +, \cdot\}$
$\mathcal{N}_5 \triangleq (\mathbb{N}, 0, S, <, +, \cdot, \mathsf{pr}, \uparrow)$ $x \uparrow y \triangleq x^y$	$\mathscr{P} = \{<, \mathsf{pr}\}$	$\mathscr{F} = \{0, S, +, \cdot, \uparrow\}$
$\mathcal{N}_6 \triangleq \cdots$		

Question: Is a new predicate (function) definable from earlier ones?

first-order definability over $\mathbb{N}$

- every number n is definable from 0 and S:

$$
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  which implies the graph of $S^{\mathcal{N}_2}$ is defined by the WFF $(x + 1 \doteq y)$.

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Hint. Use the following equivalence for all $m, n, p \in \mathbb{N}$
$(p = 0) \vee (p = m + n)$ iff
$(m \cdot p + 1) \cdot (n \cdot p + 1) = p^2 \cdot (m \cdot n + 1) + 1$

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  YES pr(n) is true iff $\varphi(n)$ is true, where $\varphi(x)$ is the WFF

  $$\varphi(x) \;\triangleq\; \neg(x \doteq 1) \;\wedge\; \forall y \forall z\, [\, (x \doteq y \cdot z) \;\rightarrow\; (y \doteq 1 \vee z \doteq 1)\,]$$

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  YES $m = n \uparrow p$ iff $\varphi(m, n, p)$ is true, where $\varphi(x, y, z)$ is the WFF . . . *(not very difficult: try it!)*

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