CS 511, Fall 2018, Handout 24 Deductive Closures and First-Order Theories

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Let Γ be a set of first-order sentences over signature Σ. The deductive closure of Γ is:

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Equivalently, a first-order theory

is the **deductive closure** of a set of first-order sentences.

the first-order theory of a relational structure

• If \mathcal{M} is a relational structure, the **first-order theory of** \mathcal{M} is:

 $\mathsf{Th}(\mathcal{M}) \triangleq \{ \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{M} \models \varphi \}$

Question: Is $Th(\mathcal{M})$ deductively closed?

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Question: Is $Th(\mathcal{M})$ deductively closed?

Yes! Can you explain why?

Consider again the structure $\mathcal{N} \triangleq (\mathbb{N}, 0, S)$ in Handout 23. The first-order theory of \mathcal{N} is:

 $\mathsf{Th}(\mathcal{N}) \triangleq \{ \varphi \mid \varphi \text{ is a first-order sentence s.t. } \mathcal{N} \models \varphi \}$

Some sentences that are true in \mathcal{N} :

S1
$$\forall x \neg (Sx \doteq 0)$$

S2
$$\forall x \forall y (Sx \doteq Sy \rightarrow x \doteq y)$$

S3
$$\forall y (\neg (y \doteq 0) \rightarrow \exists x (y \doteq Sx))$$

$$S4.1 \quad \forall x \neg (Sx \doteq x)$$

$$\mathsf{S4.2} \quad \forall x \,\neg (SSx \doteq x)$$

. . .

. . .

S4.n
$$\forall x \neg (\underbrace{S \cdots S}_{n} x \doteq x)$$

• let $\Gamma = \{$ S1, S2, S3, S4.1, S4.2, S4.3, ... $\}$

► clearly
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- in fact, the equality holds:

 $\overline{\Gamma} = \operatorname{Th}(\mathcal{N})$ (not shown here)

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▶ we therefore say that Γ is an **axiomatization** of Th(\mathcal{N}) because every sentence φ made true by \mathcal{N} is formally deduced from Γ

first-order theories of several structures over domain ${\mathbb N}$

From Handout 23:

$$\begin{split} \mathcal{N} &\triangleq (\mathbb{N}, 0, S), \quad \mathcal{N}_1 \triangleq (\mathbb{N}, 0, S, <), \quad \mathcal{N}_2 \triangleq (\mathbb{N}, 0, S, <, +) \\ \mathcal{N}_3 &\triangleq (\mathbb{N}, 0, S, <, +, \cdot) \\ \mathcal{N}_4 &\triangleq (\mathbb{N}, 0, S, <, +, \cdot, \mathsf{pr}) \quad \text{where } \mathsf{pr}(x) \triangleq \mathsf{true iff} \ x \text{ is prime} \\ \mathcal{N}_5 &\triangleq (\mathbb{N}, 0, S, <, +, \cdot, \mathsf{pr}, \uparrow) \quad \text{where } x \uparrow y \triangleq x^y \end{split}$$

1. FACT

The first-order theory of each of $\mathcal{N},$ $\mathcal{N}_1,$ and $\mathcal{N}_2,$ is axiomatizable and decidable.

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The first-order theory of each of \mathcal{N} , \mathcal{N}_1 , and \mathcal{N}_2 , is **axiomatizable** and **decidable**.

2. FACT

The first-order theory of each of \mathcal{N}_3 , \mathcal{N}_4 , and \mathcal{N}_5 , is **axiomatizable** but **not** decidable.

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