CS 511 Formal Methods, Fall 2018	Instructor: Assaf Kfoury
Lecture 10	
October 4^{th}, 2018	Laura Greige

(These lecture notes are **not** proofread and proof-checked by the instructor.)

Previously :

- Syntax
- Formal Proofs

Today :

- Semantics

1 Predicate Logic : Semantics (Handout 17)

1.1 Definition

Let $\mathcal{F}$ be a set of function symbols and $\mathcal{P}$ a set of predicate symbols. A structure $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of :

- a non-empty set A, i.e. the domain
- for every 0-ary $c \in \mathcal{F}$, a concrete element $c^{\mathcal{M}}$
- for every n-ary $f \in \mathcal{F}$, a concrete function $f^{\mathcal{M}} : A^n \to A$
- for every n-ary $P \in \mathcal{P}$, a concrete predicate $P^{\mathcal{M}} \subseteq A^n$

1.2 Examples

Consider the following models.

- s : unary function symbol for "successor", $\mathcal{M} = (\mathbb{N}, =, s^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}})$
- binary function symbol, $\mathcal{M} = (\mathbb{N}, =, s^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, +^{\mathbb{N}})$

Suppose we have the previous model and the closed formula $(\forall x \exists y \cdot s(y) \doteq x)$. Take $x = 0$, is there a y such that $s(y) = x$? **No.**

1.3 Semantic entailment, semantic validity and atisfiability

1. WFF φ is satisfiable iif there is some $\mathcal{M}$ and some ℓ such that $\mathcal{M}, \ell \vDash \varphi$

   $\to$ Let $\varphi \triangleq (\exists y \cdot s(x) \doteq y)$. Consider $\ell(x) = 3$. We have $\mathcal{M}, \ell \vDash \varphi$ therefore ϕ satisfiable.

2. WFF φ is semantically valid iif for every $\mathcal{M}$ and every ℓ, we have $\mathcal{M}, \ell \vDash \varphi$

   $\to$ Is φ defined previously valid ? No. Counter examples in next lecture.

2 Examples of First-Order Theories (Handout 18)

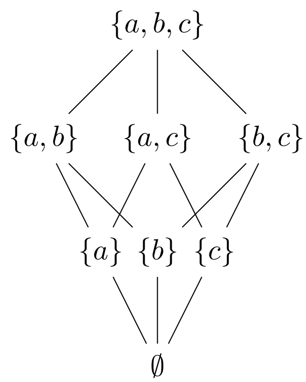
2.1 Equality vs. Equivalence

1. $\forall x, y$, if x is the same element as y, then $s(x) \doteq s(y)$ i.e. $x \doteq y \to s(x) \doteq s(y)$

2. $\forall x_1, x_2, y_1, y_2$, if $x_1 \doteq y_1 \wedge x_2 \doteq y_2 \to f(x_1, x_2) \doteq f(y_1, y_2)$

2.2 Orders

- **Partial Ordering** : Let A be the set $A = \{a, b, c\}$.



  "$\leq$" : $\subseteq$ is an example of a partial order that is not total ($\{a\}$ not related to $\{b, c\}$).

- **Dense Ordering** : You can always choose two elements and one in between.

- **Well Ordering** : Every non-empty subset has a smallest element.